Undergraduate Research Projects

Note: Purple text refers to stuff we've explicitly discussed in class.

Pursuit Games

The original cops & robbers paper [14] describes the original version of the game and characterizes cop-win (undirected) graphs. It is a bit terse, so we recommend that you start with Hahn's survey [8] instead (there are some open problems at the end of this paper, too). **The** cops & robbers book [3] is also an immensely useful resource for anyone considering research in this field.

- Come up with pursuit variant & analyze (characterization; capture time; cop number?; etc.)

  Example: Hunter & Mole.

  Hunter can't see mole but can move to any vertex at each step, while mole is constrained by the graph but can see the hunter; the goal is to characterize graphs on which hunter is guaranteed to win in bounded time (given any mole move sequence). With one hunter and one mole, the characterization turns out to be pretty neat (see chapter 4 of [11]): hunter-win $\iff$ lobster!

  This is just an example. The possibilities for games to consider are endless! Change up restrictions/assumptions/etc. on both players. I think this would be the most fun project, but it also requires the most initiative.

- **Full information pursuit games**
  - **Characterize cop-win digraphs.** No characterization is known! Maybe that means this is really hard, but even if a full characterization may not happen, some good stuff can come out of this. There is an algorithmic characterization [9] of sorts hanging about, but perhaps we find a structural characterization.
  - **Characterize cop-win graph pairs.** Meaning, cop and robber play on two different edge sets. There are some bounds on cop number in this game where the edge sets are complementary (see section 8.4 in [3] for specifics). Again, no characterization or capture time work.
  - **Distance-k cops and robbers.** Cops can capture robber from a distance. So when k=0, this is the original game. Work has been done on finding the cop-number in this game [1] for general k but as far as I know, no work on characterizations or capture time.
  - **Capture time in k-cop-win graphs.** Solved for k=1 (see [2, 7]) and an algorithmic characterization is given in [9], but I don't think much is known except the obvious upper bound of n^{k+1}. It may be quite worthwhile to work on improving this upper bound.
  - **Cops & robbers on irreflexive graphs.** In the original game, the cop and robber may remain at their current vertex at each step. So we can think of the game as being played on a reflexive graph with the players moving from vertex to adjacent vertex at each step. So what if we don't have loops, or only loops on some of the vertices? (See [4] for some thoughts on why this is indeed quite different from the original game.)
  - **Cops & robbers if cop can't revisit a vertex.** It was conjectured that under optimal play on a cop-win graph, a cop will never have to revisit a vertex. This turns out to be wrong [4](!) as there exists a graph where under optimal play, a cop must revisit a vertex. So can we characterize cop-win graphs under the condition that the cop can't revisit a vertex? Can we get an interesting bound on capture time (n is a rather trivial bound since there are only n vertices for the cop to visit)?

- **Cops & robbers if cop can't increase distance.** It was also conjectured that under optimal play on a cop-win graph, the distance between the players is monotonically decreasing. This turns out to be false as well (see, again, [4])! So if we restrict our attention to graphs on which distance is monotonically decreasing, what can we say?

- **Partial information pursuit games**
  - **Cop vs. invisible drunk.** Expected capture time is totally open. Strongly suspected to be linear but who knows. (It's $n + o(n)$ in the visible case, and embarrassingly enough, may still be $n + o(n)$ in the invisible case for all we know.) This would be a project better suited to somebody with some background in probability and/or random walks.
  - **Hunters vs. mole.** Characterize hunter-win graphs in the hunter & mole game (see above) with 2 hunters.

    Bonus: find an optimal hunter strategy (which would also answer the capture time question).

    Check out chapter 4 of this thesis; in particular, the appendix at the end of the chapter has some Matlab code that prints all the hunter-win strategies for a graph, and this code can be altered for the 2-hunter game. `http://www.math.cmu.edu/~nkomarov/NKThesis.pdf`
  - **Cop vs. invisible robber.** As far as I know, there's no characterization of cop-win graphs in the partial information case. Similarly, I can't find much evidence of capture time work besides [6], which has a conjecture on bounds, and in which the rules are more complicated than in cops & robbers. I think this might be doable on trees.
  - **Cop vs. unknown gambler.** What happens if the robber (here, a "gambler") picks a probability distribution on the vertices of G and sticks to it for the entire game? Turns out [12] that if the distribution is known, the expected capture time is exactly n on any graph. So far all that is known about the unknown distribution case is that the expected capture time is strictly between n and $2n$, and there is a conjecture that the actual upper bound is $3n/2$, the star being the worst case. This may be solvable by somebody with a bit of background in probability.

    > As a smaller sub-project, maybe we can answer this question: on what graphs does it give no advantage to the cop to know the distribution? That is, on what graphs can the cop capture the unknown gambler in time n (on K_n, for instance, the cop can check each vertex uniformly at random, so far all we know that it's strictly greater than n on C_n [11] and on the star $K_{1,n-1}$ [12], but from this we should be able to extend to a lot of different graphs with a little bit of thought).

    Link to paper: `http://arxiv.org/pdf/1308.4715.pdf`
  - **Cop with some info about robber's positions.** In all these variants, we probably want to play with the robber seeing cop's position at all times, but whatever we can make progress on is great:
    * Peter Doyle suggested a version where the robber leaves clues – e.g. the cop can see how many times each edge has been traveled, but not when.
    * Cop can "ping" the robber at each step to determine current distance. There's an elementary version of this on a grid with a great GUI online: `http://nrich.maths.org/6288`.
    * Cop knows where robber was k moves ago at every step.
    * Robber has to reveal his position every k steps.
  - Analyze Scotland Yard (the Milton Bradley board game from the 80's: see its Wikipedia article for a little more information). Recent work [13] analyzes Monte Carlo tree search as a cop strategy. (This game is basically cops & robbers but with some neat variations, like edges that only the robber can use, edges on which the cops can move faster, robber's position is only revealed five times during the course of the game, etc.)

Classical Graph Theory Problems

- **Keller's Conjecture**. This cube tiling conjecture is open only in dimension 7. It might be possible to automate Perron's case analysis from dimension 6 [15, 16]. John Mackey has some ideas on how to do this. `http://en.wikipedia.org/wiki/Keller's_conjecture`

- **Moore graphs.** Does there exist a Moore graph of girth 5 and degree 57?

  Such a graph will be strongly regular. I have developed some identities that must hold in all graphs that might be applied to get some constraints on the construction of such a graph. Then, following the argument that establishes the uniqueness of the Hoffman-Singleton graph [10], we could attempt to construct or disprove the existence of such a graph.

  Also, Noam Elkies sent John Mackey some ideas on how such a graph might be constructed and we could try to follow his outline.

  `http://en.wikipedia.org/wiki/Moore_graph`

  Hoffman & Singleton paper: `http://ieeexplore.ieee.org/stamp/stamp.jsp?tp=&arnumber=5392463`

- **Graham's number.** This number is between 13 and 2 triple-up-arrow 6. Misha Lavrov, Mitchell Lee and I obtained the current upper bound and along the way found a problem that could be a test case for the difficulty of the larger problem. The computer can actually show the answer to be 6, but the best we can do without a computer is, well, almost as big as $2 \uparrow\uparrow\uparrow 6$. Could we get the computer to show us how not to use it?

  `http://en.wikipedia.org/wiki/Graham's_number`

- **Correspondence between tournaments and graphs.** It is a bijection, actually. Given a tournament on $[n]$, you get a graph by putting an edge $\iff$ the arrow points from small to large. From the perspective of Ramsey theory, this interests me, because, for example, K_5's and I_5's are the same sub-tournament (source-sink) in this setting. This symmetry might make it easier to search for Ramsey good graphs.

- **Is there a strongly regular R(5,5)-good graph on 45 vertices?** Probably not, but perhaps by proving this we can get some insight on constructions with less constraints.

  `http://en.wikipedia.org/wiki/Ramsey_theory`

- **Seymour's 2^{nd} Neighborhood Conjecture.** Natasha and I worked on this long ago, along with countless other people. Maybe we could examine the special case of complete tripartite graphs. For more see [5] and the following two links:

  `http://www.math.uiuc.edu/~west/openp/2ndnbhd.html`

  `http://www.msri.org/people/staff/levy/files/Inv/2-4/involve-v2-n4-p02-p.pdf`

- **Heawood Vertex Characters and the 4CT.** Given a planar triangulation T, an assignment of $0, 1$ and -1 to the vertices is called **satisfiable** if there is a way to assign 1 and -1 to the triangles such that the sum of the triangles on every vertex (modulo 3) is the value assigned to the vertex.

  The 4CT corresponds to assigning 0 to every vertex. By using techniques in coding theory and group theory, perhaps one could prove a stronger result. Planar triangulations are efficiently generated by plantri.c, and David Kosbie has some results on this.

  `http://books.google.com/books?id=HMKs3OhH8OUC&pg=PA122`

- **Crossing Numbers.** There are old conjectures concerning the crossing numbers of complete and complete bipartite graphs. Interestingly, in the case of the complete graph, the truth of the conjecture for odd n implies the truth of the conjecture for $n+1$. The induction "almost works", so there are a couple of ways to address this.v

  One can consider the special case in which an n-gon is drawn, and the rest of the edges are to be drawn in without crossing the n-gon. The conjectured optimum can be obtained in this way, and I conjecture that it is uniquely obtained up to rotation and reflection for even n. Daniel Kane showed a neat lower bound for the number of crossings in drawings of this type and we hoped that we could extrapolate to general drawings somehow.

  One could also look at a slightly different class of graphs, like the complete graph minus a perfect matching, trying to make an induction argument work. Perhaps weighting the edges to yield different penalties for different crossings would yield a unique minimizer that could be verified by induction.

  For more see:

  `http://en.wikipedia.org/wiki/Crossing_number_(graph_theory)`

  `http://www3.ntu.edu.sg/home/guohua/richter-thomass.pdf`

  `http://www.math.uwaterloo.ca/~brichter/pubs/June2008.pdf`

  Daniel Kane's preprint: `http://www.math.cmu.edu/~jmackey/kane_crossing.pdf`

- **Computing local coverings without face-sharing.** If you're interested in coding something, here is a small project of interest that gives a flavor of what is necessary going forward.

  Consider strings of length n that use the numbers $-1, 0$, and 1. These correspond to centers of cubes that touch, but do not cover, the origin. Say that a pair of strings are **adjacent** if in some coordinate one of them is 1 while the other is -1 and they also differ in another coordinate (could be 0 and 1, 1 and -1, just differ).

  For example, $(-1, 1, 0, 1)$ is adjacent to $(1, 1, 1, 1)$, while $(1, 0, 0, 0)$ is not adjacent to $(-1, 0, 0, 0)$.

  The weight of a string is $2^{\text{number of zeros in the string}}$.

  For example, the weight of $(-1, 1, 0, 1)$ is 2 while the weight of $(1, 0, 0, 0)$ is 8.

  A set of strings such that every pair are adjacent and the sum of the weight is $2^{\text{length of the strings}}$, is called a **good covering**.

  I would like to know how many good coverings there are in dimensions 3, 4 and 5, for example. All good coverings in dimension 3 look essentially like (up to permutation of the columns and multiplying columns by ± 1) the following set:

  $(1, 1, 1)$: weight 1 $(-1, -1, -1)$: weight 1 $(0, -1, 1)$: weight 2 $(1, 0, -1)$: weight 2 $(-1, 1, 0)$: weight 2

  All pairs are adjacent and the sum of the weights is $8 = 2^3$.

  Is anyone interested in coding this?

- **Graceful tree conjecture**. A labeling of vertices with labels $\ell(v)$ from $[m] \cup \{0\}$ is a **graceful labeling** if $\{|\ell(v) - \ell(u)||u, v \in V(G)\} = [m]$. A graph is **graceful** if it has a graceful labeling. A lot of work has been done trying to characterize what graphs are graceful, but a long-outstanding conjecture posits that all trees are graceful. In particular, whether lobsters are graceful remains open.

  Other possible directions include: (1) characterizing hard-to-label graceful graphs – i.e. those that have a unique graceful labeling, (2) looking at nearly-graceful labellings – labellings that come close but fail slightly, and maybe (3) labelings that are graceful over some other set $S \neq [m]$ with $|S| = m$.

  This dynamic survey contains a (four page!) list of the status of the problem for different graphs and graph classes: `http://www.combinatorics.org/ojs/index.php/eljc/article/viewFile/DS6/pdf/`

This is an undergrad honors thesis providing an extensive and very readable survey of graceful labeling results: `http://math.stanford.edu/theses/Robeva%20Honors%20Thesis.pdf`

This is a masters thesis which surveys graceful tree results, specifically: `http://www.math.sc.edu/~czabarka/Graceful.pdf`

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