DEPARTMENT OF MATHEMATICAL SCIENCES
CARNEGIE MELLON UNIVERSITY

Math 21-259 Calculus in 3D
Practice Final Exam Solutions

1. (15 points) Find symmetric equations for the line of intersection L of the two planes $x+y+z=1$ and $x-2y+3z=1$. Also, find the angle between these two planes.

   Solution: To find the equation of the line, we need a point and a direction vector. Note that the direction vector of the required line is $\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2$ where $\hat{\mathbf{n}}_1 = \langle 1, 1, 1\rangle$ and $\hat{\mathbf{n}}_1 = \langle 1, -2, 3\rangle$ are normal vectors of the given planes. Thus, the direction vector $=$ $\langle 5, -2, -3\rangle$.
   To find a point on L, we can find the point where the line intersects the xy-plane by setting $z=0$ in the equations of both planes. By solving the given equations of planes simultaneously after setting $z=0$, we get $x=1$ and $y=0$. Thus, the required line passes through a point (1, 0, 0) and hence the equation of the line is given by

$$\frac{x-1}{5} = \frac{y}{-2} = \frac{z}{-3}.$$

2. (15 points) Let $\mathbf{r}(t) = (\sqrt{2}t, e^t, e^{-t})$.

   (a) Calculate the arc length function $s(t)$ measured from t = 0.

   Solution: We have,

$$\|\mathbf{r}'(t)\| = \sqrt{(\sqrt{2})^2 + (e^t)^2 + (-e^{-t})^2} = \sqrt{2+e^{2t}+e^{-2t}} = \sqrt{(e^t+e^{-t})^2} = e^t + e^{-t}.$$

   We have, $s(t) = \int_0^t \|\mathbf{r}'(u)\|\, du = \int_0^t e^u + e^{-u}\, du = e^t - e^{-t}$.

   (b) Find the equation of the line tangent to the curve at the point $\mathbf{r}(1)$.
   Solution: The tangent line is given by

$$\mathbf{R}(t) = \mathbf{r}(1) + t\mathbf{r}'(1) = (\sqrt{2}, e, e^{-1}) + t(\sqrt{2}, e, -e^{-1}).$$

   (c) Compute the unit tangent vector $\hat{\mathbf{T}}(t)$.

   Solution : $\hat{\mathbf{T}}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{(\sqrt{2}, e^t, -e^{-t})}{e^t+e^{-t}}$.

(d) Compute $\kappa(t)$.

Solution : We have $\mathbf{r}''(t) = (0, e^t, e^{-t})$. So

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{2} & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{pmatrix} = (2, -\sqrt{2}e^{-t}, \sqrt{2}e^t).$$

So we get,

$$\kappa(t) = \frac{\sqrt{4 + 2e^{-2t} + 2e^{2t}}}{(e^t + e^{-t})^3} = \frac{\sqrt{2}\sqrt{2 + e^{-2t} + e^{2t}}}{(e^t + e^{-t})^3} = \frac{\sqrt{2}(e^t + e^{-t})}{(e^t + e^{-t})^3} = \frac{\sqrt{2}}{(e^t + e^{-t})^2}.$$

3. (10 points) Find all local maximum, local minimum, and saddle points of $f(x, y) = e^{4y - x^2 - y^2}$.

Solution: First we find all the partial derivatives of f up to the second order.

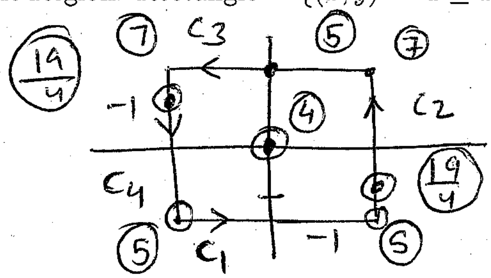
$$\begin{aligned} f_x &= -2xe^{4y-x^2-y^2} \\ f_y &= (4 - 2y)e^{4y-x^2-y^2} \\ f_{xx} &= (4x^2 - 2)e^{4y-x^2-y^2} \\ f_{xy} &= -2x(4 - 2y)e^{4y-x^2-y^2} \\ f_{yy} &= (4y^2 - 16y + 14)e^{4y-x^2-y^2} \end{aligned}$$

Next, we set $f_x = 0 = f_y$ which implies $x = 0$ and $y = 2$, so the only critical point is (0, 2). Recall that $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2$. Note that $D(0, 2) = 4e^8 > 0$ and $A = f_{xx}(0, 2) = -2e^4 < 0$, so $f(0, 2) = e^4$ is a local minimum.

4. (20 points) Find the absolute maximum and minimum values of $f(x, y) = x^2 + y^2 + x^2y + 4$ on the set $D = \{(x, y) : -1 \leq x \leq 1,\ -1 \leq y \leq 1\}$. Also, give the points at which the function attains its maximum and minimum values.

Solution: We are given that $f(x, y) = x^2 + y^2 + x^2y + 4$.

Step 1. Sketch the Region: Rectangle $= \{(x, y) : -1 \leq x \leq 1,\ -1 \leq y \leq 1\}$.



Step 2. Find critical points in the interior of the domain.

$$\nabla f =< 2x + 2xy, 2y + x^2 >= \mathbf{0} \Rightarrow 2x(y+1) = 0 \text{ and } x^2 = -y \Rightarrow x = 0,\ y = -1.$$

Note that $y = -1$ is on the boundary of the given domain, so the only critical point is (0, 0). Label it and record the value of $f(0,0) = 4$.

Step 3. Find the extreme values on the boundary of the domain.

Parametrize C_1**:** $x(t) = -1 + 2t,\ y(t) = -1,\ 0 \le t \le 1$.
Here $f_1(t) = f(x(t), y(t)) = (-1+2t)^2 + 1 - (-1+2t)^2 + 4 = 5$, constant.

Parametrize C_2**:** $x(t) = 1,\ y(t) = -1 + 2t,\ 0 \le t \le 1$.
Here $f_2(t) = f(x(t), y(t)) = 1 + (-1+2t)^2 + (-1+2t) + 4 = 4t^2 - 2t + 5$.
Note $f_2'(t) = 0 \Rightarrow t = 1/4$.
$f_2(1/4) = f(1, -1/2) = 19/4,\ f_2(0) = f(1,-1) = 5,\ f_2(1) = f(1,1) = 7$.

Parametrize C_3**:** $x(t) = 1 - 2t,\ y(t) = 1,\ 0 \le t \le 1$.
Here $f_3(t) = f(x(t), y(t)) = (1-2t)^2 + 1 + (1-2t)^2 + 4 = 2(1-2t)^2 + 5$.
Note $f_3'(t) = -8(1-2t) = 0\ \Rightarrow t = 1/2$.
$f_3(0) = f(1,1) = 7,\ f_3(1) = f(-1,1) = 7,\ f_3(1/2) = f(0,1) = 5$.

Parametrize C_4**:** $x(t) = -1,\ y(t) = 1 - 2t,\ 0 \le t \le 1$.
Here $f_4(t) = f(x(t), y(t)) = 1 + (1-2t)^2 - (1-2t) + 4$.
Note $f_4'(t) = -4(1-2t) + 2 = 8t - 2 = 0\ \Rightarrow t = 1/4$.
$f_4(0) = f(-1,1) = 7,\ f_4(1) = f(-1,1) = 0,\ f_4(1/4) = f(-1,1/2) = 19/4$.

Absolute Maximum = 7 at $(\pm 1, 1)$
Absolute Minimum = 4 at (0, 0).

5. (20 points) Find the dimensions of the rectangular box of maximum volume if the total surface area is given as 64 cm^2 using the method of Lagrange multipliers.

Solution: Let x, y, z denote the length, width and the height of the rectangular box respectively. Note that the Surface area of the rectangular box is given by $S = 2(xy + yz + zx) = 64$ cm^2. Note $z = \frac{32-xy}{x+y}$.

Maximize the Volume function, $f(x, y) = xy\frac{32-xy}{x+y}$. Then $f_x(x, y) = \frac{32y^2-2xy^3-x^2y^2}{(x+y)^2} = y^2\frac{(32-2xy-y^2)}{(x+y)^2}$ and $f_y(x, y) = x^2\frac{(32-2xy-x^2)}{(x+y)^2}$. Set $f_x = 0$ and $f_y = 0$ which implies

$$32 - 2xy - x^2 = 0 \text{ and } 32 - 2\text{xy} - \text{y}^2 = 0.$$

You may now use your best way to solve these equations simultaneously. One way to do so is by setting $32 - 2xy - x^2 = 32 - 2xy - y^2 = 0$. This implies that $x^2 = y^2 \Rightarrow x = y$ since both x and y are positive. By Substituting $x = y$ in any of the two equations above, we get $32 - 2x^2 - x^2 = 0 \Rightarrow x^2 = 32/3$. Thus, $x = y = 4\sqrt{\frac{2}{3}}$ and $z = 4\sqrt{\frac{2}{3}}$. Thus the box is a cube with edges $4\sqrt{\frac{2}{3}}$.

6. (15 points) Let $T = \{(x, y, z) : 0 \leq z \leq 6, z/2 \leq x \leq 3, x \leq y \leq 6 - y\}$ be the solid in space. Set up(not compute) a triple integral in the order $dxdydz$ that gives the volume of the solid T.

Solution: This problem is same as the part (b) of the tetrahedron problem in Exam 3. Look for its solution in Exam 3 solution packet.

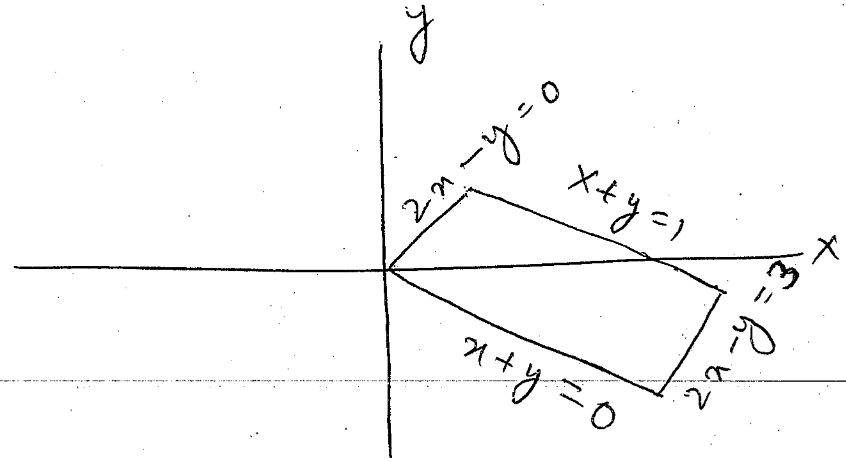
7. (15 points) Find the volume of the solid in the first octant which is bounded by the cone $x^2+y^2 = 3z^2$, by the planes $x = 0$ and $x = \sqrt{3}y$, and by the sphere $4x^2+4y^2+4z^2 = 1$.

Solution: Note that the equation of the sphere can be written as $\rho = \frac{1}{2}$ and equation of the cone as $\phi = \frac{\pi}{3}$ in spherical coordinates. From this it follows that $0 \leq \rho \leq 1/2$ and $0 \leq \phi \leq \pi/3$. Also, observe that the equation of the planes $x = 0$ and $x = \sqrt{3}y$ indicates that $0 \leq \theta \leq \pi/6$. Thus, we see that the volume of the solid is given by

$$\int_0^{\pi/6} \int_0^{\pi/3} \int_0^{1/2} \rho^2 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\phi \, \mathrm{d}\theta = \left[\frac{(\rho)^3}{3}\right]_0^{1/2} [-\cos\phi]_0^{\pi/3} \frac{\pi}{6} = \frac{\pi}{12}\frac{(1/2)^3}{3} = \frac{\pi}{288}.$$

8. (20 points) Evaluate $\iint_R (x+y)\cos\pi(2x^2+xy-y^2)\,dx\,dy$ where R is the parallelogram with vertices (0, 0), (1, -1), (1/3, 2/3), and (4/3, -1/3).

Solution: Note that R bounded by lines $x+y=0,\ x+y=1,\ 2x-y=0$ and $2x-y=3$.



This suggests that we choose $u=x+y,\ v=2x-y$. Thus, the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}=-3$ and hence the given integral can be written as

$$\begin{aligned}\iint_R (x+y)\cos\pi(2x^2+xy-y^2)\,dx\,dy &= \int_0^1\int_0^3 (u)\cos\pi uv\frac{1}{3}\,dv\,du\\ &= \frac{1}{3}\int_0^1\left[u\frac{1}{\pi u}\sin\pi uv\right]_0^3 du\\ &= \frac{1}{3}\int_0^1\frac{1}{\pi}\sin 3\pi u\,du\\ &= \frac{1}{3}\left[\frac{1}{3(\pi)^2}\cos 3\pi u\right]_0^1 = -\frac{2}{9(\pi)^2}.\end{aligned}$$

9. (a) (15 points) Determine whether or not the vector field $\mathbf{F}(x,y,z)=<yz,xz,xy>$ is conservative. If it is then f such that $\nabla f=\mathbf{F}$.

Solution: We are given that $P=yz,\ Q=xz,\ R=xy$. We need to verify that $R_y=Q_z,\ P_z=R_x$, and $Q_x=P_y$. Indeed they are all equal which implies that $\mathbf{F}$ is a gradient field, that is, $\mathbf{F}=\nabla f$.

In order to find f, we need to solve $f_x=yz, f_y=xz$, and $f_z=xy$. Then $f(x,y,z)=xyz+g(y,z)$ and $f_y=xz+g_y$. But we are given that $f_y=xz=xz+g_y\Rightarrow g_y=0\Rightarrow g(y,z)=h(z)$. Now, we use the third equation to get $xy+h'(z)=xy\ \Rightarrow h(z)=C$. Thus, we see that $f(x,y,z)=xyz+C$.

(b) (5 points) Compute $\int_C \mathbf{h}.\mathbf{dr}$ where C is given by $\mathbf{r}(t) = (t\cos t)\mathbf{i} + (t^2\cos t + 3\sin^5(t))\mathbf{j}$, $0 \le t \le \pi/2$.

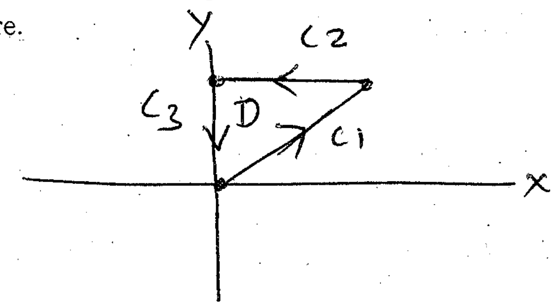
Solution: By Fundamental Theorem of line integral,

$$\int_C \mathbf{h}.\mathbf{dr} = f(\mathbf{r}(\pi/2)) - f(\mathbf{r}(0)) = f(0,3,0) - f(0,0,0) = 0.$$

10. (20 points) Compute the line integral of the vector field $\mathbf{F}(x,y) =< xy, x^2y >$ over the boundary of the triangle with vertices (0, 0), (0, 1), (2, 1) directly **and** by using Green's theorem.

Solution:

(a) Directly: First we sketch the triangle and label each side of it by a curve as shown below in the figure.



Note that the required line integral $\int_C \mathbf{F}.\mathbf{dr} = \int_{C_1} \mathbf{F}.\mathbf{dr} + \int_{C_2} \mathbf{F}.\mathbf{dr} + \int_{C_3} \mathbf{F}.\mathbf{dr}$.

On C_1, $x = 2y$ and $0 \le y \le 1$. Note,

$$\begin{aligned}\int_{C_1} \mathbf{F}.\mathbf{dr} &= \int_{C_1} xy\,dx + x^2y\,dy = \int_0^1 2y(y)\,2dy + (2y)^2y\,dy \\ &= \int_0^1 4y^2 + 4y^3\,dy = \left[\frac{4}{3}y^3 + y^4\right]_0^1 = \frac{7}{3}.\end{aligned}$$

On C_2, $y = 1$ and x is from 2 to 0, so this implies that $dy = 0$. Note,

$$int_{C_2}\mathbf{F}.\mathbf{dr} = \int_{C_2} xy\,dx + x^2y\,dy = \int_2^0 x\,dx = \left[\frac{x^2}{2}\right]_2^0 = -2.$$

On C_3, $x = 0$ and y is from 1 to 0, so this implies that $dx = 0$. Note,

$$\int_{C_3} \mathbf{F}.\mathbf{dr} = \int_{C_3} xy\,dx + x^2y\,dy = 0.$$

This implies that $\int_C \mathbf{F}.\mathbf{dr} = \frac{7}{3} - 2 = \frac{1}{3}$.

(b) Green's Theorem: We have that

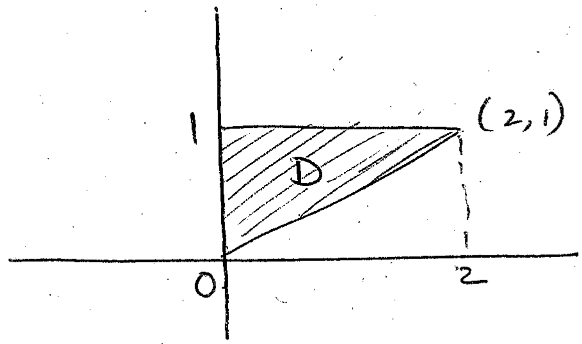
$$\begin{aligned}
\int_C \mathbf{F}.\mathrm{dr} &= \iint_D (Q_x - P_y)\,\mathrm{d}A \\
&= \iint_D (2xy - x)\,\mathrm{d}A = \int_0^2 \int_{x/2}^1 (2xy - x)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^2 \left[xy^2 - xy\right]_{y=x/2}^{y=1} \mathrm{d}x \\
&= \int_0^2 (\frac{x^2}{2} - \frac{x^3}{4}\,\mathrm{d}x \\
&= \left[\frac{x^3}{6} - \frac{x^4}{16}\right]_0^2 \\
&= \frac{8}{6} - \frac{16}{16} = \frac{1}{3}.
\end{aligned}$$

11. (10 points) Find the surface area of the part of the surface $z = 1 + 3x + 2y^2$ that lies above the triangle with vertices (0, 0), (0, 1), and (2, 1).

Solution: To find the surface area, we use

$$A(S) = \iint_D \sqrt{1 + (g_x)^2 + (g_y)^2}\,\mathrm{d}A$$

where $g(x, y) = 1 + 3x + 2y^2$ and D is the region enclosed by the given triangle as shown below.



Thus,

$$\begin{aligned}
A(S) &= \int_0^1 \int_0^{2y} \sqrt{1 + 9 + 16y^2}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1 2y\sqrt{10 + 16y^2}\,\mathrm{d}y \\
&\overset{u=10+16y^2}{=} \int_{10}^{26} \sqrt{u}\frac{1}{16}\,\mathrm{d}u = \frac{1}{24}[26^{3/2} - 10^{3/2}].
\end{aligned}$$

12. (20 points) Evaluate the surface integral $\iint_S \text{curl}\mathbf{F}.d\mathbf{S}$, where $\mathbf{F}(x, y, z) = -xy\mathbf{j} - xz\mathbf{k}$ and S consists of the paraboloid $z = x^2+y^2$, $0 \le z \le 1$, and the disk $x^2+y^2 \le 1$, $z = 1$ by following different ways:

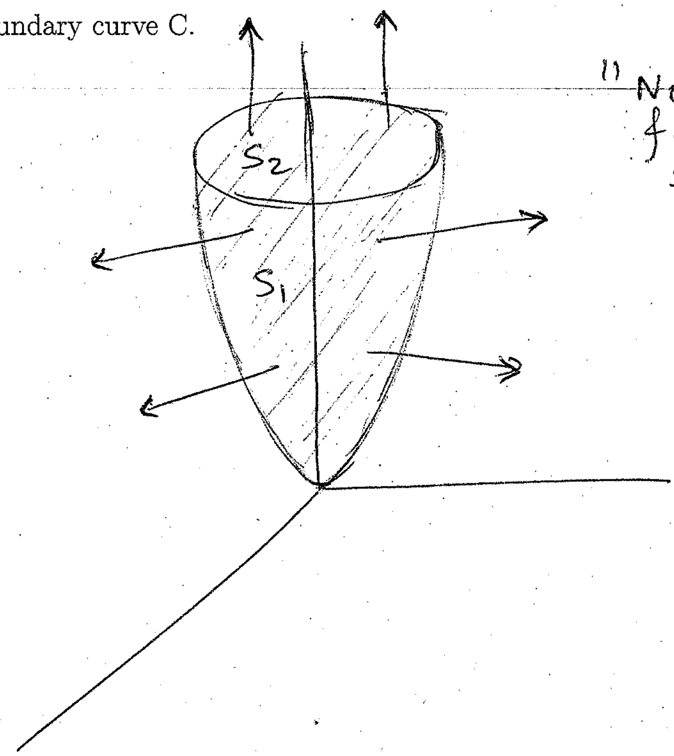
(a) Directly.

Note that $\text{curl}\mathbf{F} = z\mathbf{j} - y\mathbf{k}$. This problems is similar to problem 11 from homework 15. Check out its solution in there and the answer should turn out to be zero.

(b) By using Stokes' Theorem: It is a bad question for using Stokes' theorem as it has no boundary. However, we can still argue by Stokes' theorem that

$$\iint_S \text{curl}\mathbf{F}.d\mathbf{S} = \int_C \mathbf{F}.d\mathbf{r} = 0$$

since there is no boundary curve C.



"No boundary for closed surfaces"