DEPARTMENT OF MATHEMATICAL SCIENCES CARNEGIE MELLON UNIVERSITY

Math 21-259 Calculus in 3D
Midterm Exam I Solution

Allowed Time: 50 mins February 04, 2011

Exam Instructions:

1. This is a closed book and closed notes exam.
2. In this exam, there are 5 questions in total.
3. The last question has few subparts that are multiple choice questions.
4. Use vector notations appropriately.
5. You should write your solution in the space provided.
6. If you need more space then you may use the back of the sheet but you should indicate it clearly.
7. To receive full credit, you should show all your work and box your final answer.
8. Calculators or computers are NOT allowed.
9. Receiving a phone call or a text message during the test will result in an automatic R.

1. Given $\mathbf{a} = \mathbf{2i} + \mathbf{3j} - \mathbf{k}$ and $\mathbf{b} = \mathbf{4i} - \mathbf{3j}$.

   (i) (5 points) Find the unit vector in the direction of the vector $\mathbf{b}$.

   Solution:
   Unit vector in the direction of $\mathbf{b} = \mathbf{u_b} = \frac{\mathbf{b}}{|\mathbf{b}|} = \frac{\mathbf{4i-3j}}{5}$.

   (ii) (8 points) Find $\text{comp}_{\mathbf{b}}\mathbf{a}$ and $\text{proj}_{\mathbf{b}}\mathbf{a}$.

   Solution:
   $\text{comp}_{\mathbf{b}}\mathbf{a} = \mathbf{a}.\mathbf{u_b} = \frac{\mathbf{1}}{\mathbf{5}}(\mathbf{2}(\mathbf{4}) + \mathbf{3}(-\mathbf{3})) = -\frac{\mathbf{1}}{\mathbf{5}}$.
   $\text{proj}_{\mathbf{b}}\mathbf{a} = \text{comp}_{\mathbf{b}}\mathbf{a}\mathbf{u_b} = -\frac{\mathbf{1}}{\mathbf{25}}\mathbf{4i} - \mathbf{3j}$.

   (iii) (5 points) Find cosine of the angle between the two vectors $\mathbf{a}$ and $\mathbf{b}$.

   Solution:
   Let θ be the angle between the vector a and b. Then
   $\cos\theta = \mathbf{u_a}.\mathbf{u_b} = \frac{\mathbf{2i+3j-k}}{\sqrt{\mathbf{14}}}.\frac{\mathbf{4i-3j}}{\mathbf{5}} = -\frac{\sqrt{\mathbf{14}}}{\mathbf{70}}$.

2. Let P(1, 1, 1), Q(0, 3, 1), R(0, 1, 4), and S(1, 0, 0) be four given points in space.

   (i) (3 points) Express the vectors **PQ**, **PR** and **PS** in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

   Solution:
   $\mathbf{PQ} = \mathbf{OQ} - \mathbf{OP} =< \mathbf{0}, \mathbf{3}, \mathbf{1} > - < \mathbf{1}, \mathbf{1}, \mathbf{1} >=< -\mathbf{1}, \mathbf{2}, \mathbf{0} >= -\mathbf{i} + \mathbf{2j}$,
   $\mathbf{PR} = \mathbf{OR} - \mathbf{OP} =< \mathbf{0}, \mathbf{1}, \mathbf{4} > - < \mathbf{1}, \mathbf{1}, \mathbf{1} >=< -\mathbf{1}, \mathbf{0}, \mathbf{3} >= -\mathbf{i} + \mathbf{3k}$,
   $\mathbf{PS} = \mathbf{OS} - \mathbf{OP} =< \mathbf{1}, \mathbf{0}, \mathbf{0} > - < \mathbf{1}, \mathbf{1}, \mathbf{1} >=< \mathbf{0}, -\mathbf{1}, -\mathbf{1} >= -\mathbf{j} - \mathbf{k}$.

   (ii) (8 points) Find the area of the triangle PQR and the volume of the parallelopiped with edges **PQ**, **PR**, and **PS**.

   Solution:
   Area of the triangle PQR:

$$\frac{1}{2}|\mathbf{PQ} \times \mathbf{PR}| = |\mathbf{6i} + \mathbf{3j} + \mathbf{2k}| = \frac{\sqrt{\mathbf{36} + \mathbf{9} + \mathbf{4}}}{\mathbf{2}} = \frac{\mathbf{7}}{\mathbf{2}}.$$

   Volume of the required parallelopiped:

$$|(\mathbf{PQ} \times \mathbf{PR}).\mathbf{PS}| = |(\mathbf{6i} + \mathbf{3j} + \mathbf{2k}).(-\mathbf{j} - \mathbf{k})| = \mathbf{5}.$$

   (iii) (4 points) Find the equation of the plane through P, Q, and R, expressed in the form of $ax + by + cz = d$.

   Solution:
   Normal for the required plane is given by:

$$\mathbf{N} = \mathbf{PQ} \times \mathbf{PR} = \mathbf{6i} + \mathbf{3j} + \mathbf{2k}$$

   If we take a point P(1, 1, 1) then the equation will be:

$$6(x - 1) + 3(y - 1) + 2(z - 1) = 0.$$

   (iv) (4 points) Is the line through (1, 2, 3) and (2, 2, 0) parallel to the plane in part (iii)? Explain why or why not.

   Solution:
   Direction vector of the line through (1, 2, 3) and (2, 2, 0) is given by: $\mathbf{d} =< \mathbf{1}, \mathbf{0}, -\mathbf{3} >$. If the line that passes through (1, 2, 3) and (2, 2, 0) is parallel to the plane then the direction vector of the line should be perpendicular to the normal of the plane. That is, $\mathbf{d}.\mathbf{N} = \mathbf{0}$. To check this, consider $\mathbf{d}.\mathbf{N} =< \mathbf{1}, \mathbf{0}, -\mathbf{3} > . < \mathbf{6}, \mathbf{3}, \mathbf{2} >= \mathbf{6} - \mathbf{6} = \mathbf{0}$. This justifies that the answer to the above question should be "yes".

3. Given a point $A(1,1,1)$ and a plane $\mathcal{P} : x - y - z = 2$.

(i) (5 points) Find the distance of the point A from the plane $\mathcal{P}$.

Solution:
Distance of the point A from the plane $\mathcal{P}$ $= \left|\frac{1-1-1-2}{\sqrt{3}}\right| = \sqrt{3}$.

(ii) (5 points) Write the scalar parametric equations of the line $\mathcal{L}$ that passes through the point A and is perpendicular to the plane $\mathcal{P}$.

Solution:
Direction vector of the required line is parallel to the normal vector of the plane. Thus, the direction vector can be taken as $\mathbf{d} =< \mathbf{1}, -\mathbf{1}, -\mathbf{1} >$.
Required scalar parametric equations: $x(t) = 1 + t,\ y(t) = 1 - t,\ z(t) = 1 - t$.

(iii) (5 points) Let B be the the point of intersection of the line $\mathcal{L}$ and the plane $\mathcal{P}$. Find B.

Solution:
To find the point of intersection of $\mathcal{L}$ and $\mathcal{P}$, we solve them simultaneously. If we plug $x = 1 + t,\ y = 1 - t,\ z = 1 - t$ in the equation of the plane, we get

$$(1 + t) - (1 - t) - (1 - t) = 2 \Rightarrow -1 + 3t = 2 \Rightarrow t = 1.$$

Thus, we get the point of intersection as $B(2,0,0)$.

(iv) (2 points) Using distance formula, find the distance between the points A and B.

Solution:
Distance between the points A and B $= \sqrt{(1-2)^2 + (1-0)^2 + (1-0)^2} = \sqrt{3}$.

4. Given a curve $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\sqrt{3}\mathbf{k}$ that represents a circular helix.

(i) (16 points) Find a unit tangent vector and a principal unit normal vector at $t = \frac{\pi}{2}$.

Solution:
Note that $\mathbf{r}'(\mathbf{t}) = -\sin \mathbf{t}\mathbf{i} + \cos \mathbf{t}\mathbf{j} + \sqrt{\mathbf{3}}\mathbf{k}$ and
$\mathbf{T}(\mathbf{t}) = \frac{\mathbf{r}'(\mathbf{t})}{||\mathbf{r}'(\mathbf{t})} = \frac{\mathbf{1}}{\mathbf{2}}(-\sin \mathbf{t}\mathbf{i} + \cos \mathbf{t}\mathbf{j} + \sqrt{\mathbf{3}}\mathbf{k})$.
Thus, the unit tangent vector is given by $\mathbf{T}(\frac{\pi}{\mathbf{2}}) = \frac{\mathbf{1}}{\mathbf{2}}(-\mathbf{i} + \sqrt{\mathbf{3}}\mathbf{k})$.

Again, note that $\mathbf{T}'(\mathbf{t}) = \frac{\mathbf{1}}{\mathbf{2}}(-\cos \mathbf{t}\mathbf{i} - \sin \mathbf{t}\mathbf{j})$ and
$\mathbf{N}(\mathbf{t}) = \frac{\mathbf{T}'(\mathbf{t})}{||\mathbf{T}'(\mathbf{t})} = -\cos \mathbf{t}\mathbf{i} - \sin \mathbf{t}\mathbf{j}$.
Thus, the principal unit normal vector is given by $\mathbf{N}(\frac{\pi}{\mathbf{2}}) = -\mathbf{j}$.

(ii) (10 points) Find an equation of the osculating plane at the indicated point($t = \frac{\pi}{2}$).

Solution:
To find the equation of the osculating plane, note that the normal vector is given by

$$\mathbf{T}(\frac{\pi}{\mathbf{2}}) \times \mathbf{N}(\frac{\pi}{\mathbf{2}}) = -\frac{\sqrt{\mathbf{3}}}{\mathbf{2}}\mathbf{i} + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{k}$$

and the point that the plane passes through is given by:

$$(\cos(\frac{\pi}{2}), \sin(\frac{\pi}{2}), \sqrt{3}\frac{\pi}{2}) = (0, 1, \frac{\sqrt{3}\pi}{2}).$$

Thus the equation of the plane is given by:

$$\frac{\sqrt{3}}{2}(x-0) + 0(y-1) - \frac{1}{2}(z - \frac{\sqrt{3}\pi}{2}) = 0$$

which is same as

$$\frac{\sqrt{3}}{2}x + \frac{1}{2}(z - \frac{\sqrt{3}\pi}{2}) = 0.$$

5. Suppose $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ be three non-zero vectors.

5(a). (5 points) If $\mathbf{a} \, . \, \mathbf{b} = \mathbf{a} \, . \, \mathbf{c}$, then

(i) $\mathbf{a}$ is perpendicular to $\mathbf{b} - \mathbf{c}$. ——- **CORRECT CHOICE**

(ii) $\mathbf{a}$ is parallel to $\mathbf{b} - \mathbf{c}$.

(iii) $\mathbf{b} = \mathbf{c}$.

(iv) I want zero for this problem.

(v) None of the above.

5(b). (5 points) If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, then

(i) $\mathbf{a}$ is perpendicular to $\mathbf{b} - \mathbf{c}$.

(ii) $\mathbf{a}$ is parallel to $\mathbf{b} - \mathbf{c}$. ——- **CORRECT CHOICE**

(iii) $\mathbf{b} = \mathbf{c}$.

(iv) I want zero for this problem.

(v) None of the above.

5(c). (5 points) If $\mathbf{a} \, . \, \mathbf{b} = \mathbf{a} \, . \, \mathbf{c}$ and $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, then

(i) $\mathbf{a}$ is perpendicular to $\mathbf{b} - \mathbf{c}$.

(ii) $\mathbf{a}$ is parallel to $\mathbf{b} - \mathbf{c}$.

(iii) $\mathbf{b} = \mathbf{c}$. ——- **CORRECT CHOICE**

(iv) I want zero for this problem.

(v) None of the above.

5(d) (5 points) Is it true that $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$? Answer with either a YES or a NO. Support your answer with a proper reasoning (if YES) and a counter example(if NO).

Solution:
Answer is NO. Note that if we take $\mathbf{a} = \mathbf{i} - \mathbf{j}, \ \mathbf{b} = \mathbf{j}, \ \mathbf{c} = \mathbf{k}$ then
L.H.S $= (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{0}$ and
R.H.S $= \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -\mathbf{k}$.
Clearly, the left hand side is not equal to the right hand side. This suggests that the equality may not hold in general.