DEPARTMENT OF MATHEMATICAL SCIENCES
CARNEGIE MELLON UNIVERSITY

Math 21-259 Calculus in 3D
Midterm Exam III
Solutions

1. (20 points) Use the method of Lagrange Multipliers to find the maximum and the minimum values of the function $F(x, y, z) = 2x + 3y + 5z$ subject to the constraint $x^2 + y^2 + z^2 = 38$, and the point(s) where these values occur.

   Solution: Here $g(x, y, z) = x^2 + y^2 + z^2 = 38 = k$.

   Lagrange Condition: $\nabla F = \lambda \nabla g$

$$\begin{aligned} 2 &= 2\lambda x & (1)\\ 3 &= 2\lambda y & (2)\\ 5 &= 2\lambda z & (3)\\ x^2 + y^2 + z^2 &= 38 & (4) \end{aligned}$$

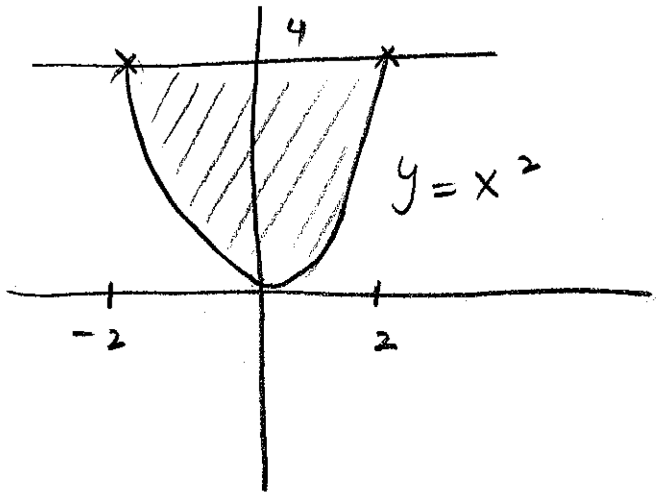
   From equations (1), (2), (3), it follows that $\lambda, x, y, z \neq 0$ and $x = \frac{1}{\lambda}$, $y = \frac{3}{2\lambda}$, $z = \frac{5}{2\lambda}$. By plugging all of these values in equation (4), we get $1 + \frac{9}{4} + \frac{25}{4} = 38\lambda^2 \Rightarrow \lambda = \pm\frac{1}{2}$. This yields us two critical points: $(2, 3, 5)$ and $(-2, -3, -5)$.

   Maximum value of F is $F(2, 3, 5) = 38$.
   Minimum value of F is $F(-2, -3, -5) = -38$.

2. Let R be the region in the x, y-plane bounded by the line $y = 4$, and the parabola $y = x^2$. Let $f(x, y) = 4xe^{y^2}$.

(a) (2 points) Sketch the region R.

Solution:



(b) (5 points) Express $\iint_R f(x, y)\,\mathrm{d}A$ as a repeated integral, integrating first with respect to y. DO NOT EVALUATE.

Solution: $\int_{-2}^{2} \int_{x^2}^{4} f(x, y)\,\mathrm{d}y\,\mathrm{d}x$

(c) (5 points) Express $\iint_R f(x, y)\,\mathrm{d}A$ as a repeated integral, integrating first with respect to x. DO NOT EVALUATE.

Solution: $\int_{0}^{4} \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y)\,\mathrm{d}x\,\mathrm{d}y$

(d) (8 points) Evaluate $\iint_R f(x, y)\,\mathrm{d}A$ by choosing an appropriate integration order.

Solution: We choose the one in which we integral first with respect to x as we don't know the antiderivative of e^{y^2}.

$$\begin{aligned}\int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} 4xe^{y^2}\,\mathrm{d}x\,\mathrm{d}y &= \int_0^4 \left[2x^2 e^{y^2}\right]_{x=-\sqrt{y}}^{x=\sqrt{y}} \mathrm{d}y \\ &= \int_0^4 2(\sqrt{y})^2 e^{y^2} - 2(-\sqrt{y})^2 e^{y^2}\,\mathrm{d}y = 0.\end{aligned}$$

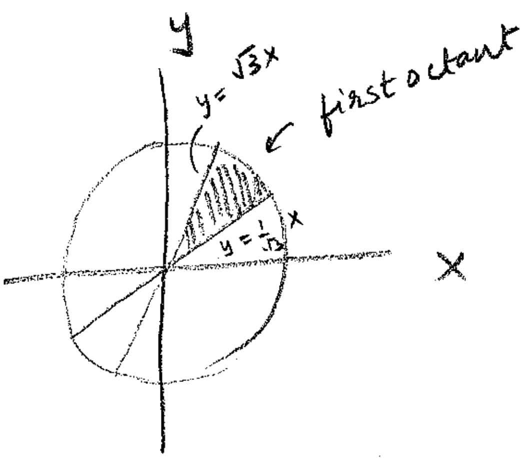
3. (20 points) Evaluate $\iiint_S \frac{1}{\sqrt{x^2+y^2+z^2}}\,\mathrm{d}V(x,y,z)$ where S is the solid in the **first octant** that lies between the cones $z=\sqrt{x^2+y^2}$ and $z=\sqrt{3x^2+3y^2}$, between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$, and between the planes $y=\sqrt{3}x$ and $\sqrt{3}y=x$.

Solution: Since the problem involves sphere and cones, it would be wise to use spherical coordinates.
First of all, we shall translate the given equations into spherical coordinates by using the following equations: $x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, $z=\rho\cos\phi$, $x^2+y^2+z^2=\rho$, $x^2+y^2=\rho^2\sin^2\phi$.

Rectangular Coordinates	Spherical Coordinates
$x^2+y^2+z^2=1$	$\rho=1$
$x^2+y^2+z^2=4$	$\rho=2$
$z=\sqrt{x^2+y^2}$	$\phi=\frac{\pi}{4}$
$z=\sqrt{3x^2+3y^2}$	$\phi=\frac{\pi}{6}$
$y=\sqrt{3}x$	$\theta=\frac{\pi}{3}$
$\sqrt{3}y=x$	$\theta=\frac{\pi}{6}$

Note that the required solid is bounded by all of the above surfaces. Thus, the the required integral can be rewritten as

$$\begin{aligned}
\iiint_S \frac{1}{\sqrt{x^2+y^2+z^2}}\,\mathrm{d}V(x,y,z) &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\int_1^2 \frac{1}{\rho}\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta \\
&= \left(\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1\,\mathrm{d}\theta\right)\left(\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\sin\phi\,\mathrm{d}\phi\right)\left(\int_1^2 \frac{1}{\rho}\rho^2\,\mathrm{d}\rho\right) \\
&= \frac{\pi}{6}\left[-\cos\phi\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}\left[\frac{\rho^2}{2}\right]_1^2 \\
&= \frac{\pi}{6}\left(\frac{\sqrt{3}-\sqrt{2}}{2}\right)(2-\frac{1}{2})=\frac{\pi}{8}\left(\frac{\sqrt{3}-\sqrt{2}}{2}\right).
\end{aligned}$$



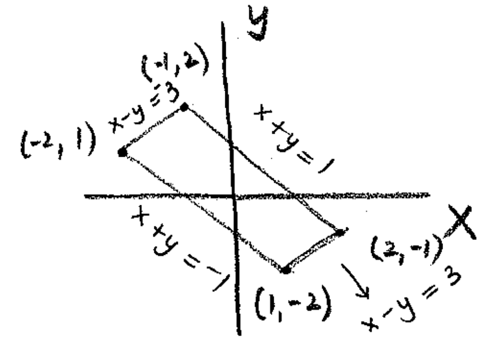
4. (20 points) Use an appropriate change of variable to evaluate

$$\iint_D (x-y)\sin(\pi(x^2-y^2))\mathrm{d}A$$

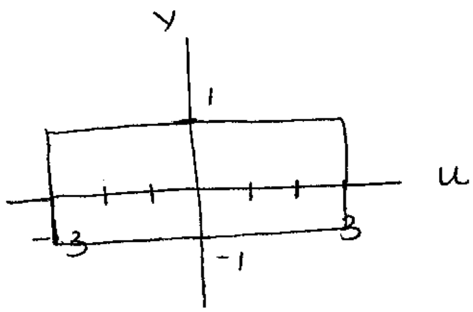
where D is the parallelogram with vertices (2, -1), (-1, 2), (-2, 1) and (1, -2).

Solution:

Step I. Sketch the given region and find the equation of each side of the parallelogram.



Step II. Use the equation of the sides to decide upon the substitution. Choose $u = x-y,\ v = x+y$ and draw the rectangle formed by the transformed points.



Step III. Find the Jacobian of the above change of variable: $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2.$

Step IV. Set up the integral in terms of u and v :

$$\iint_D (x-y)\sin(\pi(x^2-y^2))\mathrm{d}A = \int_{-3}^{3}\int_{-1}^{1} u \sin \pi uv(\frac{1}{2})\, \mathrm{d}v\, \mathrm{d}u$$

(Bad Idea to set it up in the other order)

$$= \int_{-3}^{3} \frac{-1}{2\pi}[\cos(\pi u) - \cos(-\pi u)]\, \mathrm{d}v = 0,$$

since $cos(-\theta) = \cos\theta$.

5. Let T be the solid tetrahedron with volume, $V = \int_0^3 \int_x^{6-x} \int_0^{2x} dz dy dx$.

(a) (5 points) SKETCH the solid T and LABEL the four points that make the vertices of this tetrahedron.

Solution: From the given information, the solid T can be described by the inequalities:

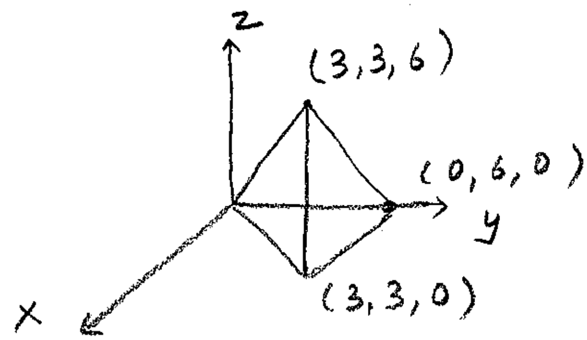
$$0 \le x \le 3,\ x \le y \le 6 - x,\ 0 \le z \le 2x.$$

I. Get the vertices of this tetrahedron:

When $x = 0$ then $y = 0$ or 6 and $z = 0$ which means (0, 0, 0) and (0, 6, 0).
When $x = 3$ then $y = 3$ and $z = 0$ or 6 which means (3, 3, 0) and (3, 3, 6).

II. Note that T is bounded by four planes: below by $z = 0$ and on the sides by $y = x, y = 6 - x,\ z = 2x$.

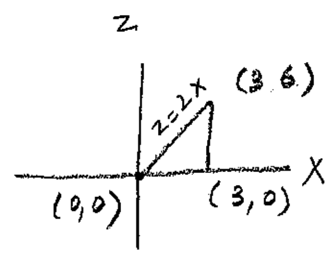
III. Sketch the solid by joining the vertices.



(b) Fill in the blanks.

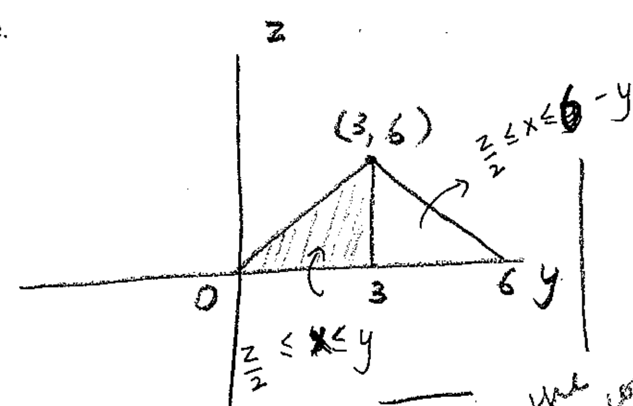
i. (6 points) $V = \int_0^6 \int_{z/2}^3 \int_x^{6-x} dy dx dz$.

Draw the projection of the solid in the xz-plane.



ii. (9 points) $V = \int_0^6 \int_{z/2}^3 \int_{z/2}^y 1\, dx dy dz + \int_0^6 \int_3^{\frac{12-z}{3}} \int_{z/2}^{6-y} 1\, dx dy dz$.

Draw the projection of the solid in yz-plane.



This is the reason we couldn't do without splitting it.

BONUS QUESTION
(5 Points)
NO PARTIAL CREDIT

Set up(not compute) double integral in polar coordinates to get the area of the region that is bounded above by the circle $x^2 + y^2 = 1$ and below by the circle $x^2 + y^2 = 2y$.

Solution:
Step I. Sketch and shade the required region.

Step II. Note that we require to split up the area into three integrals. Area $= \int_0^{\pi/6} \int_0^{2\sin\theta} r \, \mathrm{d}r \, \mathrm{d}\theta + \int_{\pi/6}^{5\pi/6} \int_0^1 r \, \mathrm{d}r \, \mathrm{d}\theta + \int_{5\pi/6}^{\pi} \int_0^{2\sin\theta} r \, \mathrm{d}r \, \mathrm{d}\theta$.

OR

A solid S, in the **first octant**, is bounded above by the paraboloid $z = 1 - x^2 - y^2$, below by the xy-plane, and the sides by the planes $y = x$, $x = 0$, and the cylinder $x^2 + y^2 = 1$. Set up a triple integral in cylindrical coordinates that gives the volume of S. DO NOT EVALUATE.
Solution:

Step I. Sketch the solid and its projection in the xy-plane.

Solution II. Volume of $S = \int_{\pi/4}^{\pi/2} \int_0^1 \int_0^{1-r^2} r \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\theta$.