DEPARTMENT OF MATHEMATICAL SCIENCES CARNEGIE MELLON UNIVERSITY

Math 21-259 Calculus in 3D
Midterm Exam II Solutions

Allowed Time: 50 mins March 21, 2011

Exam Instructions:

1. This is a closed book and closed notes exam.
2. In this exam, there are 6 questions in total including one bonus question worth 5 points.
3. Use vector notations appropriately.
4. You should write your solution in the space provided.
5. If you need more space then you may use the back of the sheet but you should indicate it clearly.
6. To receive full credit, you should show all your work and box your final answer.
7. Calculators or computers are NOT allowed.
8. Receiving a phone call or a text message during the test will result in an automatic R.

1. Let $f(x,y) = \frac{y^4}{x^4+3y^4}$.

(i) (2 points) What is the domain of f?

Solution: Domain $= \{(x,y)|(x,y) \neq (0,0)\}$.

(ii) (14 points) Find the following limits if they exist. Give proper reasoning for your answer.

(a) $\lim_{(x,y)\to(0,1)} f(x,y)$

Solution: At (0, 1), f is a continuous function being a rational function whose domain does contain (0, 1). This means that

$$\lim_{(x,y)\to(0,1)} f(x,y) = f(0,1) = 1/3.$$

(b) $\lim_{(x,y)\to(0,0)} f(x,y)$

Solution: If we let $(x,y) \to (0,0)$ along $x = 0$ then

$$\lim_{(x,y)\to(0,0)x=0} f(x,y) = \lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{y^4}{3y^4} = \frac{1}{3}.$$

If we let $(x,y) \to (0,0)$ along $y = 0$ then

$$\lim_{(x,y)\to(0,0)y=0} f(x,y) = \lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{0}{0+y^4} = 0.$$

Note, the limits along two different paths are not equal which implies that the given limit does not exist.

(iii) (4 points) Give the names of the following quadric surfaces:

(a) $z^2 = x^2 + 4y^2 + 1$

Solution: Hyperboloid of two sheets.

(b) $z = x^2 + 4y^2 + 1$.

Solution: Elliptic Paraboloid.

2. Interpret $\mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j}$ as the position of a moving object at time t.

(i) (8 points) Find the curvature of the curve $\mathbf{r}(t)$ for all t using calculus.

Solution: Here $x(t) = \sin t$ and $y(t) = \cos t$.

$$\begin{aligned}\kappa(t) &= \frac{|x''(t)y'(t) - x'(t)y''(t)|}{\sqrt{(x'(t))^2 + (y'(t))^2}} \\ &= \frac{|-\sin t(-\sin t) - \cos t(-\cos t)|}{\sqrt{\cos^2 t + (-\sin t)^2}} = 1.\end{aligned}$$

(ii) (8 points) Find the arc length of the curve $\mathbf{r}(t)$ starting from the point (0, 1) to (1, 0) using calculus.

Solution:
Initial: $x(t) = \sin t = 0$ and $y(t) = \cos t = 1 \Rightarrow t = 0$.
Final : $x(t) = \sin t = 1$ and $y(t) = \cos t = 0 \Rightarrow t = \pi/2$.

$$L(C) = \int_0^{\pi/2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = \int_0^{\pi/2} 1\, dt = \frac{\pi}{2}.$$

3. Given $f(x,y) = 2xy^2 - 2\frac{y}{x}$ and the point P : (1, 0). The gradient of f is:

$$\nabla f = (2y^2 + \frac{2y}{x^2})\mathbf{i} + (4xy - \frac{2}{x})\mathbf{j}.$$

(i) (8 points) Calculate the directional derivative of f at the point P in the direction of the vector $v = 3\mathbf{i} - 4\mathbf{j}$.

Solution: Note, $\hat{\mathbf{v}} = \frac{3\hat{\mathbf{i}} - 4\hat{\mathbf{j}}}{\sqrt{3^2+4^2}} = \frac{3}{5}\hat{\mathbf{i}} - \frac{4}{5}\hat{\mathbf{j}}$.
The directional derivative of f in the direction of $\hat{\mathbf{v}}$ at (1, 0) is given by

$$D_{\hat{\mathbf{v}}} f(1,0) = \nabla f(1,0).\hat{\mathbf{v}} = <0,-2> . <\frac{3}{5}, -\frac{4}{5}> = \frac{8}{5}.$$

(ii) (6 points) Determine a unit vector in the direction of the maximum directional derivative of f at P.

Solution: Here, $\hat{\mathbf{u}} = \frac{\nabla f(1,0)}{|\nabla f(1,0)|} = \frac{<0,-2>}{2} = <0,-1>$.

(iii) (5 points) Determine an equation for the tangent plane to the surface $z = f(x,y)$ at the point (1, 0, 0) on the surface.

Solution: The tangent plane equation is given by

$$\begin{aligned} z - 0 &= f_x(1,0)(x-1) + f_y(1,0)(y-0) = 0(x-1) - 2(y-0) \\ \Rightarrow z &= -2y. \end{aligned}$$

(iv) (5 points) Find an approximate value of $2(1.1)(0.1)^2 - 2\frac{(0.1)}{1.1}$ using part (iii).

Solution: Note that $2(1.1)(0.1)^2 - 2\frac{(0.1)}{1.1} = f(1.1, 0.1)$.
Approximate value of $f(1.1, 0.1) = -2(0.1) = -0.2$

4. (i) (10 points) Let $f(x, y) = x^4 - 2x^2 + y^2 - 2$. Find all the critical points of f.

Solution: Note, $\nabla f =< 4x^3 - 4x, 2y >$ which further implies that

$$4x^3 - 4x = 0 \quad \text{and} \quad y = 0.$$

Thus, by factoring we get that $x = 0, \pm 1$ and $y = 0$.
Critical Points: (0, 0), (1, 0), and (-1, 0).

(ii) (10 points) The points P_1 : (0, 0) and P_2 : (2, 2) are critical points for the function

$$g(x, y) = x^3 - 6xy + y^3.$$

Determine whether g has a local maximum, a local minimum or a saddle point at P_1 and P_2.

Solution: Note, $g_{xx} = 6x, g_{xy} = -6$, and $g_{yy} = 6y$.

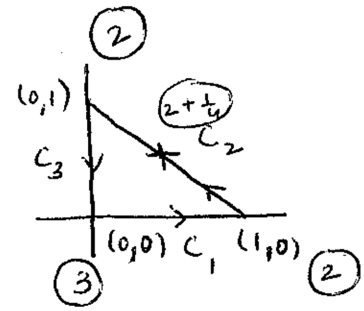
At (0, 0): $A = 0, B = -6, C = 0, \; D = AC - B^2 = -(-6)^2 < 0$. This implies that (0, 0) is a saddle point.

At (2, 2): $A = 12, B = -6, C = 12, \; D = AC - B^2 = (12)^2 - (-6)^2 > 0$. This implies that (2, 2) is a point of local minimum.

5. (20 points) Determine the absolute maximum and the absolute minimum of

$$f(x,y) = 3 + xy - x - y$$

on the set D, where D is the closed triangular region with vertices (0, 0), (1, 0), and (0, 1). Also, SKETCH the region D and label critical points on it.



Solution:

Step 1. Sketch the Region.

Step 2. Find critical points in the interior of the domain.

$$\nabla f = < y-1, x-1 > = \mathbf{0} \Rightarrow y = 1, x = 1.$$

Label it!

We shall ignore the point (1, 1) for calculation purposes since it does not belong to the domain. So, the conclusion is that there are no critical points in the interior of the domain.

Step 3. Find the extreme values on the boundary of the domain.

Parametrize C_1**:** $x(t) = t,\ y(t) = 0,\ 0 \le t \le 1.$
Here $f_1(t) = f(x(t), y(t)) = 3 - t.$
Note $f_1'(t) \neq 0$, No critical points!
$f_1(0) = f(0,0) = 3,\ f_1(1) = f(1,0) = 2.$

Parametrize C_2**:** $x(t) = 1 - t,\ y(t) = t,\ 0 \le t \le 1.$
Here $f_2(t) = f(x(t), y(t)) = 3 + (1-t)t - (1-t) - t.$
Note $f_2'(t) = (1-t) - t = 0 \Rightarrow t = 1/2.$
$f_2(0) = f(1,0) = 2,\ f_2(1) = f(0,1) = 2$, and $f_2(1/2) = f(1/2, 1/2) = 2 + 1/4.$

Parametrize C_3**:** $x(t) = 0,\ y(t) = 1 - t,\ 0 \le t \le 1.$
Here $f_3(t) = f(x(t), y(t)) = 3 - (1 - t).$
Note $f_3'(t) \neq 0$, No critical points!
$f_3(0) = f(0,1) = 2, f_3(1) = f(0,0) = 3.$

Absolute Maximum = 3 at (0, 0)
Absolute Minimum = 2 at (1, 0) and (0, 1).

BONUS QUESTION
(5 Points)

ATTEMPT EITHER OF THE FOLLOWING QUESTIONS. NO PARTIAL CREDIT!

Let f be a function of x and y with everywhere continuous second partials. Is it possible that $\frac{\partial f}{\partial x} = 2x + y$ and $\frac{\partial f}{\partial y} = 2x - y$? Give a proper reasoning to support your answer.

Solution: Given the hypothesis that f has everywhere continuous second order partial derivatives, we have by Clairaut's Theorem that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$. But we get a contradiction since $\frac{\partial^2 f}{\partial x \partial y} = 2$ and $\frac{\partial^2 f}{\partial y \partial x} = 1$ which are not equal. Hence it is not possible to have a function of two variables so that $\frac{\partial f}{\partial x} = 2x + y$ and $\frac{\partial f}{\partial y} = 2x - y$.

OR

Identify and sketch the level curves of the function $f(x, y) = \frac{y}{x^2}$ for the values $c = -1, 1$.

Level curves for the given function are given by equations of the types $y = cx^2$ where c is a constant.

For $c \neq 0$: Level curves are parabolas.

For $c = 0$: Level curve turn out to an x - axis.