Proof of the Second Derivative Test

Second Derivative Test

Suppose (a, b) is a critical point and all the second partial derivatives are continuous in a nbd of (a, b).

Let $A = f_{xx}(a, b)$, $B = f_{xy}(a, b)$, $C = f_{yy}(a, b)$ and define

$$D = AC - B^2.$$

1(i). If $D > 0$ and $A > 0$ then $f(a, b)$ is a local minimum.

1(ii). If $D > 0$ and $A < 0$ then $f(a, b)$ is a local maximum.

2. If $D < 0$ then $f(a, b)$ is neither a local maximum or minimum. (Saddle)

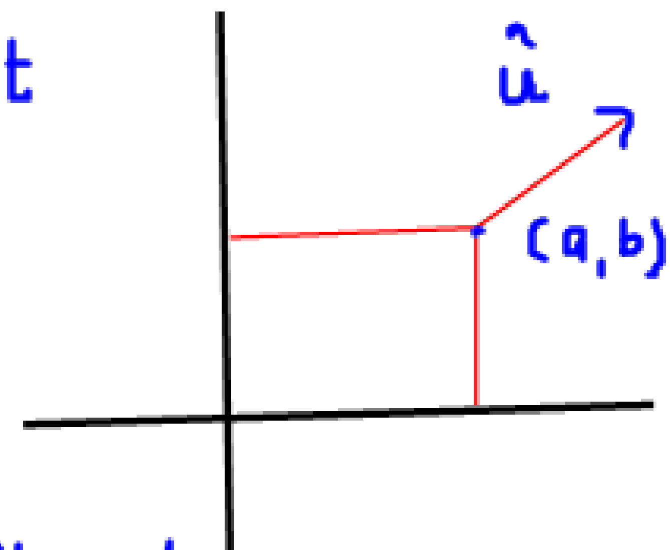
3. If $D = 0$ then the test fails.

Proof: Given: $D = AC - B^2 > 0$ and $A > 0$

WTS: $f(a,b)$ is a local minimum.

$\Leftrightarrow D^2_{\hat{u}} f \big|_{(a,b)} > 0$ for all unit vector $\hat{u}$.

Let $\hat{u} = \langle p, q \rangle$ be an unit vector.



$$D_{\hat{u}} f = \nabla f \cdot \hat{u}$$

$$= f_x p + f_y q \rightarrow \text{fn of } x \text{ and } y$$

$$D^2_{\hat{u}} f = D_{\hat{u}} (D_{\hat{u}} f) = \nabla(D_{\hat{u}} f) \cdot \hat{u}$$

$$= \langle f_{xx} p + f_{yx} q, f_{xy} p + f_{yy} q \rangle \cdot \langle p, q \rangle$$

$$D^2_{\hat{u}} f = f_{xx}p^2 + f_{yx}pq + f_{xy}pq + f_{yy}q^2$$

$$D^2_{\hat{u}} f = f_{xx}p^2 + 2f_{xy}pq + f_{yy}q^2$$

$$D^2_{\hat{u}} f(a,b) = f_{xx}(a,b)p^2 + 2f_{xy}(a,b)pq + f_{yy}(a,b)q^2$$

$$\begin{aligned} D^2_{\hat{u}} f(a,b) &= Ap^2 + 2Bpq + Cq^2 \\ &= A\left(p^2 + 2\frac{B}{A}pq + \frac{C}{A}q^2\right) \quad (A \neq 0) \\ &= A\left(\underbrace{p^2 + \frac{2B}{A}pq + \left(\frac{B}{A}q\right)^2}\; - \left(\frac{B}{A}q\right)^2 + \frac{C}{A}q^2\right) \\ &= A\left[\left(p + \frac{B}{A}q\right)^2 + q^2\left(-\frac{B^2}{A^2} + \frac{C}{A}\right)\right] \end{aligned}$$

$$D_u^2 f(a,b) = A\left(p+\frac{B}{A}q\right)^2 + A\left(-\frac{B^2}{A^2}+\frac{C}{A}\right)$$

$$= A\left(p+\frac{B}{A}q\right)^2 + \left(-\frac{B^2}{A}+C\right)$$

$$= A\left(p+\frac{B}{A}q\right)^2 + \left(\frac{\overbrace{-B^2+AC}^{D}}{A}\right)$$

$$= A\left(p+\frac{B}{A}q\right)^2 + \frac{D}{A},$$

We are given that $A>0$ and $D>0$, this implies that $D_u^2 f(a,b) > 0$.

This completes the proof of $\underline{\underline{1(i)}}$.