DEPARTMENT OF MATHEMATICAL SCIENCES
CARNEGIE MELLON UNIVERSITY

Math 21-259 Calculus in 3D
Practice Problem from Chapter 13

What sections to expect from Chapter 13 in exam?
Sections 13.1 - 13.8.

In particular, you should be able to answer the following questions.

1. How to Recognize/Sketch Vector Fields?
2. How to check if the given vector field is a gradient field or not? If a gradient field then how to find a potential function?
3. How to compute Line Integrals?
   (a) Directly.
   (b) Using Fundamental Theorem of Line integrals.
   (c) Using Green's theorem (using double integral).
   (d) Stokes' Theorem (using surface integral).
4. Application of Green's Theorem in computing area of bounded regions.
5. Verify Green's theorem.
6. How to compute Surface Area?
7. How to compute Surface Integrals?
   (a) Directly
   (b) Using Stokes' theorem.
8. Verify Stokes Theorem.

Practice Problems including Homework Problems

1. Evaluate the line integral $\int_C (y/x)\,ds$, $C: x = t^4, y = t^3$, $1/2 \le t \le 1$. (HW)

   Solution: Check out Homework solutions.

2. Evaluate the line integral $\int_C xe^y\,dx$, C is the arc of the curve $x = e^y$ from (1, 0) to (e, 1).

$$\vec{r}(t) = e^t \hat{i} + t\hat{j}, \quad 0 \le t \le 1$$

$$\int_C x e^y\,dy = \int_0^1 e^t e^t \underbrace{e^t\,dt}_{dy}$$

$$= \int_0^1 e^{3t}\,dt = \left[\frac{e^{3t}}{3}\right]_0^1$$

$$= \boxed{\frac{e^3 - 1}{3}}$$

Ans

3. Evaluate the line integral $\int_C \sin x\,dx + \cos y\,dy$, C consists of the top half of the circle $x^2 + y^2 = 1$ from (1, 0) to (-1, 0) and the line segments from (-1, 0) to (-2, 3).(HW)

   Solution: Check out Homework solutions.

4. Find the work done done by the force field $\mathbf{F}(x, y) = x \sin y\mathbf{i} + y\mathbf{j}$ on a particle that moves along the parabola $y = x^2$ from (-1, 1) to (2, 4). (HW)

   Solution: Check out Homework solutions.

5. An object, acted on by various forces, moves along the parabola $y = 3x^2$ from the origin to the point (1, 3). One of the forces acting on the object is $\mathbf{F}(x, y) = x^3\mathbf{i} + y\mathbf{j}$. Calculate the work done by $\mathbf{F}$.

$$P(x,y) = x^3 \text{ and } Q(x,y) = y$$

Note: $P_y = 0 = Q_x \Rightarrow \vec{F}$ is a gradient field.

Find f so that $\vec{F} = \nabla f$.

$$f_x = x^3 \text{ and } f_y = y$$

$$\Rightarrow f(x,y) = \frac{x^4}{4} + h(y) \text{ and } f_y = h'(y)$$

But $f_y = y = h'(y) \Rightarrow h(y) = \frac{y^2}{2} + C$

$$\Rightarrow \boxed{f(x,y) = \frac{x^4}{4} + \frac{y^2}{2} + C}$$

Workdone by $\vec{F}$
$= f(1,3) - f(0,0)$
$= 19/4$

6. Determine the work done by the force $\mathbf{F}(x, y, z) = xy\mathbf{i} + 2\mathbf{j} + 4z\mathbf{k}$ along the circular helix $C : \mathbf{r}(u) = \cos u\mathbf{i} + \sin u\mathbf{j} + u\mathbf{k}$, from $u = 0$ to $u = 2\pi$.

$$\text{Work done by } \vec{F} = \int_0^{2\pi} \vec{F}(\vec{r}(u)) \cdot \vec{r}\,'(u)\, du$$

$$= \int_0^{2\pi} \langle \cos u \sin u, 2, 4u \rangle \cdot \langle -\sin u, \cos u, 1 \rangle\, du$$

$$= \int_0^{2\pi} -\cos u \sin^2 u + 2\cos u + 4u \; du$$

$$= \left[-\frac{\sin^3 u}{3} + 2\sin u + 2u^2 \right]_0^{2\pi}$$

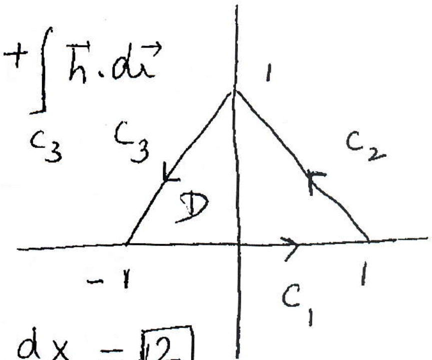
3

$$= \boxed{8\pi^2} \text{ Ans}$$

7. Evaluate the line integral $\int_C \mathbf{h}.d\mathbf{r}$ if $\mathbf{h}(x,y) = e^y\mathbf{i} - \sin \pi x\mathbf{j}$ and C is the triangle with vertices (1, 0), (0, 1), (-1, 0) traversed counterclockwise.

We can either do it directly or by using Green's theorem

(a) Directly : $\int_C \vec{h}\cdot d\vec{r} = \int_{C_1} \vec{h}\cdot d\vec{r} + \int_{C_2} \vec{h}\cdot d\vec{r} + \int_{C_3} \vec{h}\cdot d\vec{r}$



On C_1: $-1 \le x \le 1$ and $y = 0$.

$$\int_{C_1} \vec{h}\cdot d\vec{r} = \int_{C_1} e^y\,dx - \sin \pi x \cancel{dy}^{\,\to 0} = \int_{-1}^{1} dx = \boxed{2}$$

On C_2: $y = 1-x$ ~~$0 \le x \le 1$~~ x is from 1 to 0.

$$\int_{C_2} \vec{h}\cdot d\vec{r} = \int_1^0 e^{1-x}\,dx + \sin \pi x\,dx = \left[-e^{1-x} - \frac{\cos \pi x}{\pi}\right]_1^0 = \cancel{e+1}$$

$$= \boxed{1 - e - \frac{2}{\pi}}$$

On C_3: $y = 1 + x$ x is from 0 to -1

$$\int_{C_3} \vec{h}\cdot d\vec{r} = \int_0^{-1} e^{1+x}\,dx - \sin \pi x\,dx = \left[e^{1+x} + \frac{\cos \pi x}{\pi}\right]_0^{-1}$$

$$= \boxed{1 - e - \frac{2}{\pi}}$$

$$\Rightarrow \int \vec{h}\cdot d\vec{r} = \boxed{4 - 2e - \frac{4}{\pi}} \quad \text{Ans}$$

(b) Green's theorem : $\int_C \vec{h}\cdot d\vec{r} = \iint_D (Q_x - P_y)\,dA = \int_0^1\int_{y-1}^{1-y} (-\pi\cos\pi x - e^y)\,dx\,dy$

$$= \int_0^1 -2\sin(\pi(1-y)) - 2(1-y)e^y\,dy = \left[-\frac{2}{\pi}\cos\pi(1-y) - 2e^y\right]_0^1 + \int_0^1 2ye^y\,dy$$

$$= \left(-\frac{2}{\pi} - 2e\right) - \left(\frac{2}{\pi} - 2\right) + \left[2ye^y - 2e^y\right]_0^1$$

$$= -\frac{4}{\pi} - 2e + 2 + 2 = \boxed{4 - 2e - \frac{4}{\pi}} \quad \text{Ans}$$

8. Integrate $\mathbf{h}(x, y, z) = \cos x\mathbf{i} + \sin y\mathbf{j} + yz\mathbf{k}$ over the indicated path:

(a) The line segment from (0, 0, 0) to (2, 3, -1).

$C: \vec{r}(t) = 2t\hat{i} + 3t\hat{j} - t\hat{k}, \quad 0 \le t \le 1$

$$\int_C \vec{h} \cdot d\vec{r} = \int_0^1 2\cos 2t + 3\sin 3t + 3t^2 \, dt$$

$$= \Big[\sin 2t - \cos 3t + t^3\Big]_0^1$$

$$= \sin 2 - \cos 3 + 1 + 1$$

$$= \boxed{2 + \sin 2 - \cos 3} \quad \text{Ans}$$

(b) $\mathbf{r}(u) = u^2\mathbf{i} - u^3\mathbf{j} + u\mathbf{k}$, u in [0, 1].

$$\int_C \vec{h} \cdot d\vec{r} = \int_0^1 2u\cos u^2 + 3u^2 \sin u^3 - u^4 \, du$$

$$= \Big[\sin u^2 - \cos u^3 - \frac{u^5}{5}\Big]_0^1$$

$$= \sin 1 - \cos 1 - \frac{1}{5} + 1$$

$$= \boxed{\frac{4}{5} + \sin 1 - \cos 1} \quad \text{Ans}$$

9. (a) Determine whether or not $\mathbf{F}(x,y) = (xy\cos xy + \sin xy)\mathbf{i} + (x^2\cos xy)\mathbf{j}$ is a conservative vector field. If it is, then a function f such that $\mathbf{F} = \nabla f$. (HW)

Solution: Check out Homework solutions. : $f(x,y) = x\sin xy + C$.

(b) Compute the line integral of $\mathbf{F}$ over the curve $C : \mathbf{r}(t) = \sqrt{t}\mathbf{i}+(1+t^3)\mathbf{j}$, $0 \le t \le 1$.

$$\int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0)) \quad \left(\text{Fundamental theorem of line Integ}\right)$$

$$= f(1,2) - f(0,1)$$

$$= \boxed{\sin 2} \ \text{Ans}.$$

(c) Compute the line integral of $\mathbf{F}$ over the curve $C : \mathbf{r}(t) = (t^2+3)\mathbf{i} + \sin(\frac{1}{2}\pi t)\mathbf{j}$, $0 \le t \le 1$.

$$\int_C \vec{F}\cdot d\vec{r} = -f(3,0) + f(4,1)$$

$$= \boxed{+4\sin 4} \quad \text{Ans}$$

10. (a) Determine whether or not $\mathbf{F}(x,y) = \frac{y^2}{1+x^2}\mathbf{i} + \frac{2y}{\cancel{\arctan x}}\mathbf{j}$ arctanx is a conservative vector field. If it is, then a function f such that $\mathbf{F} = \nabla f$. (HW)

Solution: Check out Homework solutions.

(b) Compute the line integral of $\mathbf{F}$ over the curve $C : \mathbf{r}(t) = t^2\mathbf{i}+2t\mathbf{j}$, $0 \le t \le 1$. (HW)

Solution: Check out Homework solutions.

11. Find the work done by the force field $\mathbf{F}(x,y) = e^{-y}\mathbf{i} - xe^{-y}\mathbf{j}$ in moving an object from P(0, 1) to Q(2, 0). (HW)

Solution: Check out Homework solutions.

12. (a) Show that $\vec{h}(x,y) = (3x^2+12xy+3y^2)\mathbf{i}+(4y^3+6x^2+6xy)\mathbf{j}$ satisfies the conditions to be a gradient.

$$P(x,y) = 3x^2+12xy+3y^2 \text{ and}$$
$$Q(x,y) = 4y^3+6x^2+6xy$$

Note: $Q_x = P_y! \Rightarrow \vec{h}$ is a gradient field.

(b) Find a function $f(x,y)$ so that $\mathbf{h} = \nabla f$.

$$f_x = 3x^2+12xy+3y^2 \text{ and } f_y = 4y^3+6x^2+6xy$$

$$\Rightarrow f(x,y) = x^3+6x^2y+3xy^2+h(y) \text{ and}$$
$$f_y = 6x^2+6xy+h'(y)$$
But $f_y = 4y^3+6x^2+6xy \Rightarrow h(y) = y^4+C$

$$\Rightarrow \boxed{f(x,y) = x^3+6x^2y+3xy^2+y^4+C}$$

(c) Let C be the curve given by $\mathbf{r}(t) = (t+\cos(t))\mathbf{i}+(\sin(t)+\cos^3(t))\mathbf{j}$, $0 \le t \le \pi/2$, and compute

$$\int_C \mathbf{h}.d\mathbf{r}.$$

$$\int_C \vec{h}\cdot d\vec{r} = f(\vec{r}(\pi/2)) - f(\vec{r}(0))$$
$$= f\left(\left(\tfrac{\pi}{2},1\right)\right) - f((1,1))$$
$$= \boxed{\frac{\pi^3}{8}} \quad \underline{\text{Ans}}$$

13. Compute $\int_C \mathbf{G}.\mathbf{dr}$, where $\mathbf{G}(x,y) = (2xy + e^x - 3)\mathbf{i} + (x^2 - y^2 + \sin y)\mathbf{j}$ and C is the ellipse $4x^2 + 9y^2 = 36$.

Note that $P = 2xy + e^x - 3$

$Q = x^2 - y^2 + \sin y$

Step I : Is G a gradient field?

$$P_y = 2x = Q_x$$

So, the answer is Yes! ☺

Step II : Is C a closed curve?

Yes.

$$\boxed{\int_C \vec{G} \cdot d\vec{r} = 0}$$

by Fundamental theorem of line Integral

14. Evaluate the line integral by two methods: (a) directly and (b) using Green's Theorem. (HW)

   (a) $\oint_C xy^2\,dx + x^3\,dy$, C is the rectangle with vertices (0, 0), (2, 0), (2, 3), and (0, 3).

      Solution: Check out Homework solutions.

   (b) $\oint_C x\,dx + y\,dy$, C consists of the line segments from (0, 1) to (0, 0) and from (0, 0) to (1, 0), and the parabola $y = 1 - x^2$ from (1, 0) to (0, 1).

      Solution: Check out Homework solutions.

15. Use Green's theorem to evaluate the following line integrals:

   (a) $\int_C xe^{-2x}\,dx + (x^4 + 2x^2y^2)\,dy$, C is the boundary of the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. (HW)

      Solution: Check out Homework solutions.

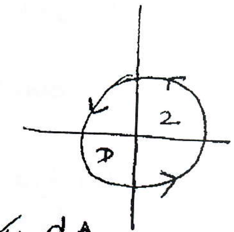
   (b) $\int_C y^2 \cos x\,dx + (x^2 + 2y \sin x)\,dy$, C is the triangle from (0, 0) to (2, 6) to (2, 0) to (0, 0). (HW)

      Solution: Check out Homework solutions.

16. Use Green's theorem to evaluate $\int_C y^2\,dx + (3x+2xy)\,dy$ where C is the circle of radius 2 oriented counterclockwise.

$P(x,y) = y^2$ and $Q(x,y) = 3x + 2xy$

$P_y = 2y$ and $Q_x = 3 + 2y$



$$\int_C y^2\,dx + (3x+2xy)\,dy \overset{G.T}{=} \iint_D (3+\cancel{2y}) - \cancel{2y}\,dA$$

$$= \iint_D 3\,dA$$

$$= 3\iint_D dA$$

$$= 3(\text{area of } D)$$

$$= 3\,\pi(2)^2 = \boxed{12\pi} \quad \underline{\text{Ans}}$$

17. Compute the integral of the function $\mathbf{F}(x,y) = (e^{x^2}+2x^2-4x+y)\mathbf{i}+(\sin y+x+3y+2)\mathbf{j}$ over the curve $C : r(\theta) = \sin\theta,\ 0 \le \theta \le \pi$.

"this is in polar coordinates"

[Note $x = r\cos\theta = \sin\theta\cos\theta$ and

$y = r\sin\theta = \sin\theta\sin\theta = \sin^2\theta$

$\vec{r}(\theta) = \sin\theta\cos\theta\,\hat{i} + \sin^2\theta\,\hat{j}$

$0 \le \theta \le \pi$]

Note that $Q_x = 1 = P_y \Rightarrow \vec{F} = \nabla f$ for some f.

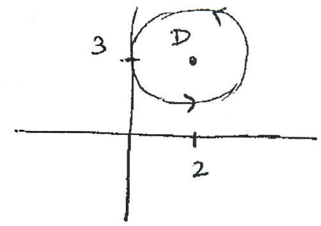
$$\int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(\pi)) - f(\vec{r}(0)) = f((0,0)) - f((0,0))$$

$$= 0 \;\ddot{\smile}$$

(Note! We didn't even require to compute f since the given curve is a closed curve.)

18. Let C be the circle of radius 2 centered at (2, 3) oriented counterclockwise.

(a) Find a parameterization of C using $0 \leq t \leq 2\pi$.



$$x(t) = 2\cos t + 2$$

$$y(t) = 2\sin t + 3$$

$$\vec{r}(t) = (2\cos t + 2)\hat{i} + (2\sin t + 3)\hat{j}, \quad 0 \leq t \leq 2\pi.$$

(b) Use your parameterization of C to express

$$I = \int_C (4xy + 2y)\, dx + (2x^2)\, dy$$

as an integral dt.

$$I = \int_0^{2\pi} \Big[\big(4(2\cos t + 2)(2\sin t + 3) + 2(2\sin t + 3)\big)(-2\sin t) + 2(2\cos t + 2)^2 (2\cos t)\Big]\, dt$$

$$= \int_0^{2\pi} \Big[(-4\sin t)(2\sin t + 3)(4\cos t + 5) + (4\cos t)(2\cos t + 2)^2\Big]\, dt$$

"Horrible calculations"

(c) Use Green's theorem to compute this integral.

$$I \overset{G.T}{=} \iint_D (4x) - (4x + 2)\, dA$$

$$= -2\,(\text{area of circle})$$

$$= -2\,\pi(2)^2 = \boxed{-8\pi} \quad \underline{\text{Ans}}$$

19. Use Green's theorem to find the area of the following bounded regions.

(a) Region bounded by the astroid $r(u) = \cos^3 ui + \sin^3 uj$, $0 \le u \le 2\pi$.

$$\text{Area} = \int_C \frac{-y\,dx + x\,dy}{2} = \int_0^{2\pi} \frac{-\sin^3 u(3\cos^2 u(-\sin u)) + \cos^3 u\,(3\sin^2 u(\cos u))}{2}\,du$$

$$= \int_0^{2\pi} \frac{3}{2}\left(\cos^2 u \sin^4 u + \cos^4 u \sin^2 u\right) du$$

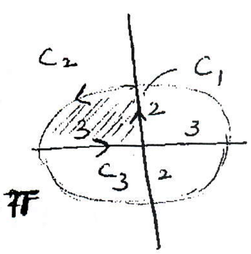
$$= \int_0^{2\pi} \frac{3}{2} \sin^2 u \cos^2 u\,du = \int_0^{2\pi} \frac{3}{8}\sin^2 2u\,du$$

$$= \int_0^{2\pi} \frac{3}{8}\,\frac{(1-\cos 4u)}{2}\,du = \frac{3}{16}\left[u - \frac{\sin 4u}{4}\right]_0^{2\pi} = \boxed{\frac{3\pi}{8}}$$

Ans

(b) Region in the second quadrant bounded by the ellipse $4x^2 + 9y^2 = 36$.

$$\text{Area} = \int_{C_1+C_2+C_3} \frac{-y\,dx + x\,dy}{2}$$



$x(t) = 3\cos t$ and $y(t) = 2\sin t$ $\frac{\pi}{2} \le t \le \pi$

On C_1: $0 \le y \le 2$ and $x = 0$.

$$\int_{C_1} \frac{-y\,dx + x\,dy}{2} = 0.$$

On C_2: $x(t) = 3\cos t$ and $y(t) = 2\sin t$, $\frac{\pi}{2} \le t \le \pi$

$$\int_{C_2} \frac{-y\,dx + x\,dy}{2} = +\int_{\pi/2}^{\pi} \frac{6\sin^2 t + 6\cos^2 t}{2}\,dt = 3\left(\frac{\pi}{2}\right)$$

On C_3: x is from -3 to 0 and $y = 0$.

$$\int_{C_3} \frac{-y\,dx + x\,dy}{2} = 0 \Rightarrow \boxed{\text{Area} = \frac{3\pi}{2}}$$

Ans

19. Use Green's theorem to find the area of the following bounded regions.

(a) Region bounded by the astroid $\mathbf{r}(u) = \cos^3 u\mathbf{i} + \sin^3 u\mathbf{j}$, $0 \le u \le 2\pi$.

$$\text{Area} = \int_C \frac{-y\,dx + x\,dy}{2} = \int_0^{2\pi} \frac{-\sin^3 u(3\cos^2 u(-\sin u))}{2}\,du + \frac{\cos^3 u\,(3\sin^2 u(\cos u))}{2}\,du$$

$$= \int_0^{2\pi} \frac{3}{2}\left(\cos^2 u \sin^4 u + \cos^4 u \sin^2 u\right) du$$

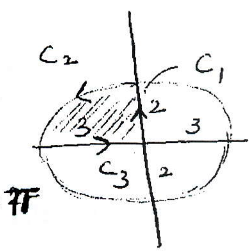
$$= \int_0^{2\pi} \frac{3}{2} \sin^2 u \cos^2 u\,du = \int_0^{2\pi} \frac{3}{8}\sin^2 2u\,du$$

$$= \int_0^{2\pi} \frac{3}{8}\frac{(1-\cos 4u)}{2}\,du = \frac{3}{16}\left[u - \frac{\sin 4u}{4}\right]_0^{2\pi} = \boxed{\frac{3\pi}{8}}$$

Ans

(b) Region in the second quadrant bounded by the ellipse $4x^2 + 9y^2 = 36$.

$$\text{Area} = \int_{C_1+C_2+C_3} \frac{-y\,dx + x\,dy}{2}$$



$x(t) = 3\cos t$ and $y(t) = 2\sin t$ $\frac{\pi}{2} \le t \le \pi$

On C_1: $0 \le y \le 2$ and $x = 0$.

$$\int_{C_1} \frac{-y\,dx + x\,dy}{2} = 0.$$

On C_2: $x(t) = 3\cos t$ and $y(t) = 2\sin t$, $\frac{\pi}{2} \le t \le \pi$

$$\int_{C_2} \frac{-y\,dx + x\,dy}{2} = +\int_{\pi/2}^{\pi} \frac{6\sin^2 t + 6\cos^2 t}{2}\,dt = 3\left(\frac{\pi}{2}\right)$$

On C_3: x is from -3 to 0 and $y = 0$.

$$\int_{C_3} \frac{-y\,dx + x\,dy}{2} = 0 \Rightarrow \boxed{\text{Area} = \frac{3\pi}{2}}$$

Ans

20. (a) Use Green's theorem to show that if is the region enclosed by a simple closed curve C, then $\oint_C (x+2y)\,dx + (3x-4y)\,dy = area(\Omega)$.

$P = x + 2y$ and $Q = 3x - 4y$

Note that: $\displaystyle\oint_C P\,dx + Q\,dy \overset{G.Th}{=} \iint_{\Omega} (Q_x - P_y)\,dA$

$\displaystyle = \iint_{\Omega} (3-2)\,dA = area(\Omega)$

(b) What is the value of the integral if

i. C is the circle: $(x-1)^2 + (y-2)^2 = 4$ and

From (a) $\displaystyle\oint_C (x+2y)\,dx + (3x-4y)\,dy$ = area of the disk of radius 2

$= \boxed{4\pi}$ Ans

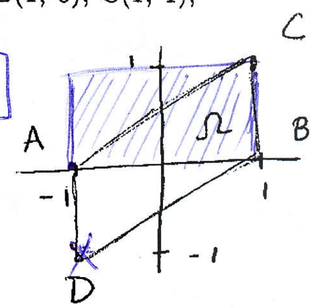
ii. C is the path formed by joining the four points: A(-1, 0), B(1, 0), C(1, 1), D(-1, 1).

From (a), the value of integral $= \boxed{2}$

$= area(\Omega)$

= area of parallelogram

$= \text{abs}\begin{vmatrix} -2 & 1 \\ 0 & 1 \end{vmatrix} \begin{matrix} \leftarrow \vec{AC} \\ \leftarrow \vec{AD} \end{matrix} = \boxed{2}$ Ans



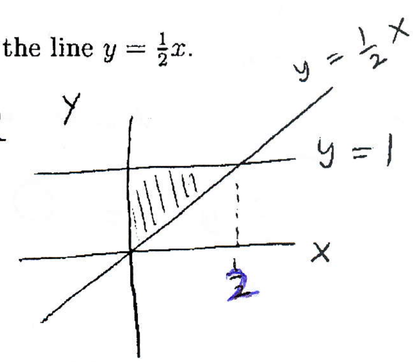
iii. C is the path formed by the y-axis, the line y = 1, and the line $y = \frac{1}{2}x$.

From (a), the value of the integral

= area of the triangle

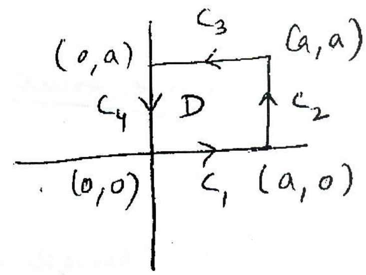
$= \frac{1}{2}(1)(2) = \boxed{\frac{4}{4} = 1}$ Ans
  base height

$= \boxed{1}$



21. Verify Green's theorem for the vector field $\mathbf{F}(x, y) = (1+10xy+y^2)\,dx + (6xy+5x^2)\,dy$ and the curve C which is the the square with vertices (0, 0), (a, 0), (a, a), (0, a).

<u>Direct Computation</u>: $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_{C_1+C_2+C_3+C_4} \vec{F}\cdot d\vec{r}$



<u>On C_1</u>: $0 \le x \le a$ and $y = 0$.

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_1} (1+10xy+y^2)\,dx + (6xy+5x^2)\,dy$$

$$= \int_0^a dx + 0 = \boxed{a}$$

<u>On C_2</u>: $x = a$ and $0 \le y \le a \Rightarrow dx = 0$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^a (6ay + 5a^2)\,dy = \left[3ay^2 + 5a^2y\right]_0^a$$

$$= 3a^3 + 5a^3 = \boxed{8a^3}$$

<u>On C_3</u>: x is from a to 0 and $y = a \Rightarrow dy = 0$.

$$\int_{C_3} \vec{F}\cdot d\vec{r} = \int_a^0 (1 + 10ax + a^2)\,dx = \left[x + 5ax^2 + a^2x\right]_a^0$$

$$= -a - 5a^3 - a^3 = -a - 6a^3$$

<u>On C_4</u>: $x = 0$ and y is from a to 0. $\Rightarrow dx = 0$.

$$\int_{C_4} \vec{F}\cdot d\vec{r} = 0 \Longrightarrow \boxed{\int_C \vec{F}\cdot d\vec{r} = 2a^3}$$

<u>Green's theorem</u>:
$$\iint_D (6y + 10x) - (10x + 2y)\,dA$$

$$= \int_0^a\!\!\int_0^a 4y\,dx\,dy = [x]_0^a\,[2y^2]_0^a$$

$$= \boxed{2a^3} \quad \text{Ans}$$

22. Find the curl and the divergence of the vector field

$$\mathbf{F}(x, y) = \frac{x}{x^2 + y^2 + z^2}\mathbf{i} + \frac{y}{x^2 + y^2 + z^2}\mathbf{j} + \frac{z}{x^2 + y^2 + z^2}\mathbf{k}.$$

(HW)

Solution: Check out Homework solutions.

23. Determine whether or not the vector field $\mathbf{F} = e^z\mathbf{i} + \mathbf{j} + xe^z\mathbf{k}$ is conservative. If it is conservative, find a function f such that $\mathbf{F} = \nabla f$. (HW)

Solution: Check out Homework solutions.

24. Find the surface area of the part of the plane $2x + 5y + z = 10$ that lies inside the cylinder $x^2 + y^2 = 9$. (HW 15)

Solution: Check out Homework solutions.

25. Find the surface area of the part of the surface $z = 1 + 3x + 2y^2$ that lies above the triangle with vertices (0, 0), (0, 1), and (2, 1).(HW 15)

Solution: Check out Homework solutions.

26. Find the surface area of the part of the hyperbolic paraboloid $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.(HW 15 - example done in class)

Solution: Check out Homework solutions or class notes.

27. Evaluate the surface integral $\iint_S \sqrt{1 + y^2 + z^2}\, dS$, where S is the helicoid with vector equation $\mathbf{r}(u, v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + v\mathbf{k}$, $0 \le u \le 1$, $0 \le v \le \pi$. (HW 15)

Solution: Check out Homework solutions.

28. Evaluate the surface integral $\iint_S xy\,dS$, where S is the triangular region with vertices (1, 0, 0), (0, 2, 0), and (0, 0, 2).(HW 15)

Solution: Check out Homework solutions.

29. Evaluate the surface integral $\iint_S yz\,dS$, where S is the part of the plane $x+y+z=1$ that lies in the first octant.(HW 15)

Solution: Check out Homework solutions.

30. Evaluate the surface integral $\iint_S xyz\,dS$, where S is the part of the sphere $x^2+y^2+z^2=1$ that lies above the cone $z=\sqrt{x^2+y^2}$. (HW 15)

Solution: Check out Homework solutions.

31. Evaluate the surface integral $\iint_S \mathbf{F}.d\mathbf{S}$, where $\mathbf{F}(x,y,z)=x\mathbf{i}-z\mathbf{j}+y\mathbf{k}$ and S is the part of the sphere $x^2+y^2+z^2=4$ in the first octant, with orientation towards the origin.(HW 15)

Solution: Check out Homework solutions.

32. Evaluate the surface integral $\iint_S \mathbf{F}.d\mathbf{S}$, where $\mathbf{F}(x,y,z)=y\mathbf{j}-z\mathbf{k}$ and S consists of the paraboloid $y=x^2+z^2$, $0\leq y\leq 1$, and the disk $x^2+z^2\leq 1$, $y=1$.(HW 15)

Solution: Check out Homework solutions.

33. Evaluate the surface integral $\iint_S \mathbf{F}.d\mathbf{S}$, where $\mathbf{F}(x,y,z)=y\mathbf{i}+(z-y)\mathbf{j}+x\mathbf{k}$ and S is the surface of the tetrahedron with vertices (0, 0, 0), (1, 0, 0), and (0, 0, 1).(HW 15)

Solution: Check out Homework solutions.

34. Use Stokes' theorem, that is, line integrals to evaluate $\iint_S \mathbf{F}.d\mathbf{S}$ where

(a) $\mathbf{F}(x,y,z) = x^2e^{yz}\mathbf{i} + y^2e^{xz}\mathbf{j} + z^2e^{xy}\mathbf{k}$ and S is the hemisphere $x^2+y^2+z^2 = 4,\ z \geq 0$,oriented upwards.(HW 15)

Solution: Check out Homework solutions.

(b) $\mathbf{F}(x,y,z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ and S is the part of the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 5$, oriented upward.(HW 15)

Solution: Check out Homework solutions.

35. Use Stokes' theorem, that is, surface integrals to evaluate $\int_C \mathbf{F}.d\mathbf{r}$ where

(a) $\mathbf{F}(x,y,z) = e^{-x}\mathbf{i} + e^x\mathbf{j} + e^z\mathbf{k}$, and C is the boundary of the part of the plane $2x + y + 2z = 2$ in the first octant oriented counterclockwise as viewed from above.(HW 15)

Solution: Check out Homework solutions.

(b) $\mathbf{F}(x,y,z) = xy\mathbf{i} + 2z\mathbf{j} + 3y\mathbf{k}$, and C is the curve of intersection of the plane $x+z = 5$ and the cylinder $x^2 + y^2 = 9$.(HW 15)

Solution: Check out Homework solutions.

36. Verify Stokes' theorem for the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + xyz\mathbf{k}$ and S is the part of the plane $2x + y + z = 2$ that lies in the first octant oriented upward.(HW 15)

Solution: Check out Homework solutions.