Exam I Review Problem Set
Math 21-259

1. Calculate the length of the line segment AB and find the midpoint where A(2, 0, -1) and B(0, -1, 1).

Length (AB) $= \sqrt{(0-2)^2 + (-1-0)^2 + (1-(-1))^2} = \sqrt{4+1+4} = \sqrt{9}$
$= 3$ Ans

Midpoint $\left(\frac{2+0}{2}, \frac{-1}{2}, \frac{-1+1}{2}\right) = \left(1, -\frac{1}{2}, 0\right)$ Ans

2. Find an equation for the plane through (-5, -2, 6) that is parallel to the xy-plane.

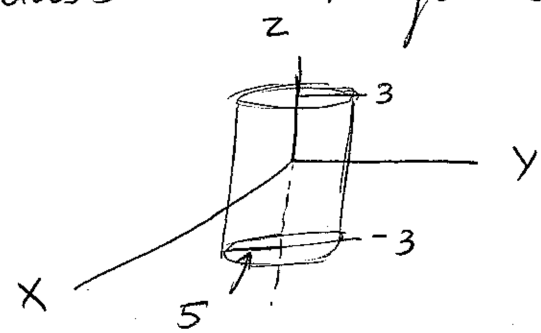
$z = \alpha$ (since plane is $\parallel^{le}$ xy-plane)
$z = 6$ (since the given plane passes through $(-5,-2,6)$)

3. Determine if the following equation represents a sphere and if so, find its center and radius: $x^2 + y^2 + z^2 + 10x + 4y - 12z + 56 = 0$.

Center $(-5,-2,6)$
Radius $= \sqrt{9} = 3$

4. Describe the region $\Omega = \{(x, y, z) : \underbrace{x^2 + y^2 \leq 25}_{\text{circle of radius 5 and center } (0,0,0)},\ -3 \leq z \leq 3\}$ in words.

Ω is a cylinder with radius 5 and height 6.



5. Find α given that $\underbrace{3\mathbf{i} + 6\mathbf{j} - 7\mathbf{k}}_{*6}$ and $\alpha\mathbf{i} - 36\mathbf{j} + 42\mathbf{k}$ are parallel.

$*6 \qquad *6 \qquad *6$

$$3 * 6 = \alpha$$

$$\Rightarrow \boxed{\alpha = 18} \text{ Ans}$$

6. Find all numbers x for which the angle between $\mathbf{c} = x\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{d} = \mathbf{i} + 1x\mathbf{j} + 3\mathbf{k}$ is $\frac{\pi}{3}$.

$$\frac{\vec{c} \cdot \vec{d}}{|\vec{c}||\vec{d}|} = \cos\theta$$

$$\Rightarrow \frac{x + x + 9}{\sqrt{x^2+10}\sqrt{x^2+10}} = \cos\theta \Rightarrow \cos\frac{\pi}{3} = \frac{2x+9}{x^2+10} = \frac{1}{2}$$

$$\Rightarrow \boxed{x = \frac{1 \pm \sqrt{33}}{2}} \text{ Ans}$$

7. Find the scalar projection of the vector $\mathbf{a} = \mathbf{i} + \sqrt{3}\mathbf{k}$ onto the vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.

$$\text{comp}_{\hat{i}}\, \vec{a} = \cancel{\text{proj}}_{\hat{i}}\, \vec{a} = 1$$

$$\cancel{\text{pro}}\ \text{comp}_{\hat{j}}\, \vec{a} = 0$$

$$\text{comp}_{\hat{k}}\, \vec{a} = \sqrt{3}. \quad \text{Ans}$$

8. Find the vector projection of the vector $\mathbf{a} = 2\,\mathbf{i}+\mathbf{j}-3\,\mathbf{k}$ onto the vector $\mathbf{b} = 4\,\mathbf{i}-2\mathbf{j}+\mathbf{k}$.

$$\text{proj}_{\vec{b}}\,\vec{a} = \left(\text{Comp}_{\vec{b}}\,\vec{a}\right) u_{\vec{b}}$$

$$= \left(\vec{a}\cdot\frac{\vec{b}}{|\vec{b}|}\right)\frac{\vec{b}}{|\vec{b}|}$$

$$= \left(\frac{3}{\sqrt{21}}\right)\frac{4\hat{i}-2\hat{j}+\hat{k}}{\sqrt{21}} = \frac{3}{21}\left(4\hat{i}-2\hat{j}+\hat{k}\right) \quad \underline{\underline{\text{Ans}}}$$

9. Calculate the following:

(a) $\mathbf{j}\ .\ (2\mathbf{i}\times 3\mathbf{k})$ $= 6\,\hat{j}\cdot(\hat{i}\times\hat{k}) = 6\hat{j}\cdot(-\hat{j})$

$$= -6\,\hat{j}\cdot\hat{j} = -6$$

(b) $(\mathbf{i}-\mathbf{j}+3\mathbf{k})\ .\ \underbrace{(2\mathbf{i}\times 3\mathbf{k})}_{-6\hat{j}}$ $= 6$ $\underline{\underline{\text{Ans}}}$

(c) $(\mathbf{i}-\mathbf{j}+3\mathbf{k})\times(2\,\mathbf{i}+\mathbf{j})$ $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & 1 & 0 \end{vmatrix} = -3\hat{i}+6\hat{j}+3\hat{k}$ $\underline{\underline{\text{Ans}}}$

(d) The volume of the parallelopiped with the given edges. $\mathbf{i}-2\mathbf{j}+\mathbf{k}$, $2\mathbf{j}-\mathbf{k}$, $\mathbf{i}+\mathbf{j}-4\mathbf{k}$.

$$\text{Volume} \overset{\text{abs}}{=} \begin{vmatrix} 1 & -2 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & -4 \end{vmatrix} = |-7+2-2| = |-7|$$

$$= 7 \quad \underline{\underline{\text{Ans}}}$$

10. A vector parametrization for the line that passes through P(2, 4, 0) and is parallel to the line $\mathbf{r}(t) = (\mathbf{i} - \mathbf{j}) + t(3\mathbf{i} - \mathbf{k})$ is given by:

(a) $\mathbf{r}(t) = 2\mathbf{i} + 4\mathbf{j} + t(\mathbf{i} - \mathbf{k})$

(b) $\mathbf{r}(t) = 2\mathbf{i} + 4\mathbf{j} - t(3\mathbf{i} - \mathbf{k})$

(c) $\mathbf{r}(t) = 2\mathbf{i} + 4\mathbf{j} + t(\mathbf{i} + \mathbf{k})$

(d) $\mathbf{r}(t) = 2\mathbf{i} + 4\mathbf{j} + t(3\mathbf{i} + \mathbf{k})$

(e) None of the above.

$\vec{d} = \langle 3, 0, -1 \rangle$

So is $\langle -3, 0, +1 \rangle$

11. Give a vector parametrization for the line that passes through P(1, 2, -4) and is parallel to the line: $2(x-1) = 3(y-2) = 6(z+4)$

(a) $\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} + t(2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k})$

(b) $2\mathbf{i} - 6\mathbf{j} + 24\mathbf{k} + t(2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k})$

(c) $\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} - t(3, \mathbf{i} + 2\mathbf{j} + \mathbf{k})$

(d) $\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} + t(3\mathbf{i} - 2\mathbf{j} - \mathbf{k})$

(e) None of the above.

12. Find a vector parametrization for the line segment that begins at (2, 7, -1) and ends at (4, 2, 3).

$$\vec{r} = 2\hat{i} + 7\hat{j} - \hat{k} + t(2\hat{i} - 5\hat{j} + 4\hat{k})$$

with $0 \le t \le 1$ Ans

$\uparrow$ (2, 7, -1) $\quad \uparrow$ (4, 2, 3)

13. Determine whether the given two lines intersect and if they do, find the point and the angle of intersection.

$$l_1 : \mathbf{r}_1(t) = \mathbf{i} + t\,\mathbf{j}, \qquad l_2 : \mathbf{r}_2(t) = \mathbf{j} + u\,(\mathbf{i} + \mathbf{j})$$

Point: $1 = u$, $t = 1 + u$ $\Rightarrow t = 2$ point $(1, 2, 0)$

Angle: $\vec{d}_1 = \hat{j}$, $d_2 = \hat{i} + \hat{j}$

$$\cos\theta = \frac{\hat{j}}{1} \cdot \frac{(\hat{i} + \hat{j})}{\sqrt{2}} = \frac{1}{\sqrt{2}} \Rightarrow \boxed{\theta = \frac{\pi}{4}}$$

14. Find the scalar parametric equation of the line that passes through P(1, 2, -4) and is parallel to the line: $l : \mathbf{r}_1(t) = \mathbf{i} + t\,\mathbf{j}$. How about the symmetric form?

$x(t) = 1$
$y(t) = 2 + t$
$z(t) = -4$

Symmetric form cannot be written in this case b/c of the zero components the direction vector $\vec{d}$.

15. Find the distance from the point P(1, 0, 2) to the line that passes through the origin and is parallel to $2\,\mathbf{i} - \mathbf{j} + 2\,\mathbf{k}$.

P_0

$$\vec{r}(t) = t(2\hat{i} - \hat{j} + 2\hat{k})$$

$$\text{Distance} = \frac{|\vec{P_0P} \times \vec{d}|}{|\vec{d}|} = \frac{|(\hat{i} + 2\hat{k}) \times (2\hat{i} - \hat{j} + 2\hat{k})|}{3}$$

$$= \frac{|2\hat{i} + 2\hat{j} - \hat{k}|}{3}$$

$$= 1 \quad \text{Ans}$$

16. Find an equation of a plane that passes through the point P(2, 3, 4) and is perpendicular to $2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.

$\vec{N} = \langle 2, -1, 2 \rangle$

Point $(2, 3, 4)$

Eqn : $2(x-2) - 1(y-3) + 2(z-4) = 0$

17. Find an equation of a plane that passes through the point P(1, 3, 1) and contains the line $l : x = t, y = t, z = -2 + t$.

I $\vec{d} = (1, 1, 1)$

II Point $P_0(0, 0, -2)$

III Vector in the plane : $\langle 1, 3, 3 \rangle = PP_0$

IV $\vec{N} = -2\hat{j} + 2\hat{k}$ $(\langle 1, 1, 1 \rangle \times \langle 1, 3, 3 \rangle) = -2\hat{j} + 2\hat{k}$

18. Find an equation of a plane that passes through the points P(1, 0, 1), Q(2, 1, 0), R(1, 1, 1).

$\vec{PQ} = \langle 1, 1, -1 \rangle$

$\vec{PR} = \langle 0, 1, 0 \rangle$

$\vec{N} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{vmatrix} = -\hat{i} + \hat{k}$

Point $P(1, 0, 1)$

$-(x-1) + (z-1) = 0 \Rightarrow \boxed{z = x}$

19. Find the angle between the planes $x - y + z - 1 = 0,\ 2x + y + z = -1$.

$\vec{N}_1 = \langle 1, -1, 1 \rangle$

$\vec{N}_2 = \langle 2, 1, 1 \rangle$

$\cos\theta = \dfrac{\vec{N}_1 \cdot \vec{N}_2}{|\vec{N}_1|\,|\vec{N}_2|}$

$= \dfrac{\sqrt{2}}{3}$

$\Rightarrow \theta = \boxed{\cos^{-1}\dfrac{\sqrt{2}}{3}}$ Ans

20. Find the distance from the point P(3, -5, 2) to the plane $8x - 2y + z = 5$.

$$\text{distance} = \frac{|8(3) - 2(-5) + 2 - 5|}{\sqrt{8^2 + (-2)^2 + 1^2}}$$

$$= \frac{3}{\sqrt{69}} \quad \underline{\underline{\text{Ans}}}$$

21. Given an equation of a plane $x - 2y + z = 5$. Which of the following is true about the points A(2, 1, 1) and B(2, -1, 1)?

(a) Both A and B lie on the same side of the plane

(b) A and B lie of the opposite sides of the plane

(c) Both the points A and B lie on the plane

(d) The point A lie on the plane and B outside the plane

(e) The point B lie on the plane and A outside the plane

22. Find the domain of the function $f(t) = t\,\mathbf{i} + \sqrt{(t+1)}\,\mathbf{j} - e^t\,\mathbf{k}$ and check for its continuity. Also, find $f'(t)$ and $\int_0^1 f(t)dt$.

→ Not a part of Exam I

domain : $[-1, \infty)$

Continuous everywhere.

$$f'(t) = \hat{i} + \frac{1}{2}(t+1)^{1/2 - 1}\hat{j} - e^t\hat{k}$$

$$\int_0^1 f(t)\,dt = \left.\frac{t^2}{2}\hat{i} + \frac{2}{3}(t+1)^{2/3}\hat{j} - e^t\hat{k}\,\right]_0^1$$

$$= \frac{1}{2}\hat{i} + \left[\frac{2^{5/2}}{3} - \frac{2}{3}\right]\hat{j} + (1-e)\hat{k}$$

$\underline{\underline{\text{Ans}}}$

23. Find $f(t)$ given that $f'(t) = \sin t\,\mathbf{i} + 3t^2\,\mathbf{j}$ and $f(0) = \mathbf{i} - \mathbf{k}$.

$$\vec{f}(t) = -\cos t\,\hat{i} + t^3\hat{j} + \vec{C}$$
$$\vec{f}(0) = \hat{i} - \hat{k} \quad \text{(given)}$$
$$= -\hat{i} + \vec{C}$$
$$\Rightarrow \vec{C} = 2\hat{i} - \hat{k}$$
$$\Rightarrow \vec{f}(t) = (2 - \cos t)\hat{i} + (t^3 - 1)\hat{j} \quad \underline{\text{Ans}}$$

24. Find $lim_{t\to 0} 3(t^2-1)\,\mathbf{i} + \cos t\,\mathbf{j} + \frac{t}{|t|}\,\mathbf{k}$.

$$\underbrace{3(t^2-1)}_{\substack{t\to 0 \\ \downarrow \\ -3}} \qquad \underbrace{\cos t}_{\substack{t\to 0 \\ \downarrow \\ 1}} \qquad \underbrace{\tfrac{t}{|t|}}_{\substack{t\to 0 \\ \downarrow \\ \text{d.n.e}}}$$

$$\lim_{t\to 0} 3(t^2-1)\hat{i} + \cos t\,\hat{j} + \frac{t}{|t|}\hat{k} \text{ does not exist.}$$

25. Find the points on the curve $\mathbf{r}(t)$ at which $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ have opposite directions.

$$\mathbf{r}(t) = 5t\,\mathbf{i} + (3 + t^2)\,\mathbf{j}$$

$$r'(t) = 5\hat{i} + 2t\hat{j}$$

$$r'(t) \parallel^{le} r(t) \Rightarrow r(t) = \alpha\, r'(t)$$

for some $\alpha < 0$. (opposite direction)

- $5t = 5\alpha \Rightarrow \alpha = t$
- $3 + t^2 = 2\alpha t \Rightarrow 3 + t^2 = 2t^2 \Rightarrow 3 = t^2$

$$\Rightarrow t = \pm\sqrt{3}$$

But $\alpha = t < 0$

$$\Rightarrow t = -\sqrt{3}$$

$$(-5\sqrt{3}, 6, 0) \quad \underline{\underline{\text{Ans}}}$$

26. Find a unit tangent vector and the principal normal vector(which is sometimes also referred as unit normal vector) to the curve $\mathbf{r}(t) = \cos(\pi t)\,\mathbf{i} + \sin(\pi t)\,\mathbf{j} + 3t\,\mathbf{k}$ at $t = 3$. Also, parametrize the tangent line and the normal line at the same indicated point. Lastly, find an equation of the osculating plane at the same point.

Unit tangent vector $= \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$

$$\hat{T}(3) = \frac{-\pi\hat{j} + 3\hat{k}}{\sqrt{\pi^2+9}}$$

Principal normal, vector $\hat{N}(3) = \dfrac{\hat{T}'(3)}{|\hat{T}'(3)|}$

$$= \hat{i}$$

Tangent line: $r'(3) \longrightarrow$ direction vector

$P(-1,0,9) \longrightarrow$ point on the line

$\vec{r}(t) = (-\pi\hat{j} + 3\hat{k})t + (-\hat{i} + 9\hat{k})$

Normal line: $\hat{T}'(3) \longrightarrow$ direction vector

$P(-1,0,9) \longrightarrow$ point on the line

$\vec{R}(u) = -\hat{i} + 9\hat{k} + u(\hat{i})$

Osculating plane $\vec{B} = \hat{N}(3) \times \hat{T}(3) \longrightarrow$ binormal vector

Point $(-1,0,9)$

$\text{comp}_{\hat{i}}\vec{B}(x+1) + \text{comp}_{\hat{k}}\vec{B}(z-9) = 0$

27. Find a point at which the following curves intersect. Also the find angle of intersection.

$$\mathbf{r}_1(t) = t\,\mathbf{i} + t^2\,\mathbf{j} - t^3\,\mathbf{k}, \qquad \mathbf{r}_2(u) = \sin 2u\,\mathbf{i} + u\cos u\,\mathbf{j} + u\,\mathbf{k}$$

This question is given just to get an idea of the steps. Curves are not very well chosen.

I Set Coefficients equal

$$t = \sin 2u$$
$$t^2 = u\cos u$$
$$-t^3 = u.$$

II Solve for t and u

III plug the value of t (or u) back in $\vec{r}(t)$ (or $\overrightarrow{R(u)}$) to the point of intersection

IV Angle:

$$\cos\theta = \frac{r_1'(t) \cdot r_2'(u)}{|r_1'(t)|\,|r_2'(u)|}$$

t-value where they intersect

u-value where they intersect

28. Find the arc length of the curve $\underbrace{2\,\mathbf{i}+t^2\,\mathbf{j}+(t-1)^2\,\mathbf{k}}_{C}$. from $t=0$ to $t=1$.

$$L(C) = \int_0^1 |r'(t)|\,dt$$

$$= \int_0^1 |2t\hat{j} + 2(t-1)\hat{k}|\,dt$$

$$= \int_0^1 \sqrt{4t^2 + 4(t-1)^2}\,dt$$

$$= 2\underbrace{\int_0^1 \sqrt{2t^2-2t+1}\,dt}_{\text{Cal II problem.}}$$

Cal II problem.
complete the square
and use trig substitution

Not a part of Exam I.

29. Find the radius of curvature of the curve $2y = x^2$.

NOT a part of Exam I

$$\text{Curvature, } K = \frac{y''}{(1+y'^2)} = \frac{x'}{(1+x^2)^{3/2}}$$

$$= \frac{1}{(1+x^2)^{3/2}}$$

$$\text{radius of curvature} = \frac{1}{\text{curvature}} = (1+x^2)^{3/2}$$

30. Find the radius of curvature of the curve $\mathbf{r}(t) = 2t\,\mathbf{i} + t^3\,\mathbf{j}$ in terms of t.

NOT a part of Exam I Parametric

$$\text{Curvature, } \kappa = \frac{|x'y'' - x''y'|}{(x'^2 + y'^2)^{3/2}}$$

$$= \frac{|2(6t) - 0|}{(4 + 9t^4)^{3/2}} = \frac{12\,|t|}{(4+9t^4)^{3/2}}$$

$$\text{Radius of Curvature, } \rho = \frac{(4+9t^4)^{3/2}}{12\,|t|} \quad \underline{\underline{\text{Ans}}}$$