Exam I Review Problem Set
Math 21-123

1. Find a formula for the general term of the following sequence:

   (a) $\{\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, ...\}$

   $$\boxed{a_k = \cos \frac{k\pi}{4}}$$

   (b) $\{2, 7, 12, 17, ...\}$

   $$\boxed{a_n = 5n - 3}$$

   (c) $\{2, 1, 2, 1, ...\}$

   $$\boxed{a_n = 2 - \left(\frac{1 + (-1)^n}{2}\right)}$$

2. Determine whether the following sequences converges or diverges. If it converges, find the limit.

(a) $a_n = n \sin \frac{1}{n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = \lim_{x\to 0} \frac{\sin x}{x} = 1 \quad \text{Ans}$$

(b) $a_n = \frac{n}{\sqrt{n^3+1}}$

$$\lim_{n\to\infty} a_n = \infty \Rightarrow a_n \text{ diverges to } +\infty.$$

(c) $a_n = (-1)^n (1 + \frac{1}{n})^n$

$a_n$ diverges since $\lim_{n\to\infty} a_{2n+1} = -e$, $\lim_{n\to\infty} a_{2n} = e$ different limits

(d) $a_n = \frac{10^n}{n!}$

For $n \geq 10$, $0 \leq a_n = \frac{10}{n} \frac{10}{n-1} \cdots \frac{10}{2} \frac{10}{1} \leq \frac{10}{n} \frac{10}{10} \cdots \frac{10}{1} \longrightarrow 0$

By squeeze theorem, $\lim_{n\to\infty} a_n = 0$ Ans

(e) $a_n = n^2 e^{-n}$

$f(x) = x^2 e^{-x}$

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{e^x} \overset{*}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{*}{=} \lim_{x\to\infty} \frac{2}{e^x}$$

$$= 0 \quad \text{Ans}$$

3. Determine whether the following sequences are monotonic or not. Give proper reasoning for your assertion.

(i) $a_n = \ln(\frac{n}{n+1})$

$$f(x) = \ln\left(\frac{x}{x+1}\right), \quad x \geq 1$$

$$f'(x) = \frac{x+1}{x}\left[\frac{(x+1)-x}{(x+1)^2}\right] = \frac{1}{x(x+1)} > 0 \text{ for } x \geq 1$$

$\Rightarrow$ $f$ is increasing

$\Rightarrow$ $\{a_n\}$ is increasing

(ii) $a_1 = 1/4, a_n = \frac{n!}{2^n},\ n \geq 2$

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{2} > 1 \quad \text{for all } n \geq 2$$

$\Rightarrow$ $a_{n+1} > a_n$ for all $n \geq 2$

Note $a_1 = \frac{1}{4} < \frac{1}{2} = a_2$

$\Rightarrow$ $a_{n+1} > a_n$ for all $n \geq 1$

$\Rightarrow$ $\{a_n\}_{n=1}^{\infty}$ is increasing.

(iii) $a_n = (3^n + 4^n)^{1/n}$

$$(3^n+4^n)^{\frac{n+1}{n}} = (3^n+4^n)^{1/n}\,(3^n+4^n)$$
$$= (3^n+4^n)^{1/n}(3^n) + (3^n+4^n)^{1/n}\, 4^n$$

Note: $(3^n+4^n)^{1/n} > (3^n)^{1/n} = 3$ and $(3^n+4^n)^{1/n} > 4$

Thus, $(3^n+4^n)^{\frac{n+1}{n}} > 3.3^n + 4.4^n = 3^{n+1} + 4^{n+1}$

$$\Rightarrow \quad (3^n+4^n)^{1/n} > (3^{n+1}+4^{n+1})^{\frac{1}{n+1}}$$

This shows that $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence

(iv) $a_n = (1+\frac{1}{n})^n$ (HINT: $(1-x)^n \geq 1-nx$ whenever $0 \leq x \leq 1$)

$$\frac{a_{n+1}}{a_n} = \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n} = \left(\frac{n(n+2)}{(n+1)^2}\right)^n \left(1+\frac{1}{n+1}\right)$$
$$= \left(1 - \frac{1}{(n+1)^2}\right)^n \left(1+\frac{1}{n+1}\right)$$
$$\geq \left(1 - \frac{n}{(n+1)^2}\right)\left(1+\frac{1}{n+1}\right) \quad \text{(Using Hint)}$$
$$= 1 + \frac{1}{(n+1)^3} > 1$$

This shows that $a_n$ is an increasing sequence.

Note: One more proof is given on pg A16 of the text.

4. Determine whether the following sequences are bounded or not. Give proper reasoning for your assertion.

(a) $a_n = (3^n + 4^n)^{1/n}$

$$3 = (3^n)^{1/n} < (3^n + 4^n)^{1/n} < (4^n + 4^n)^{1/n} = 2^{1/n} \cdot 4 \leq 8$$

$\Rightarrow$ sequence is bounded.

(b) $a_n = \frac{n}{e^n}$

$e^n > n$ for all $n \geq 1$

$\Rightarrow$ $0 \leq a_n < 1$ for all $n \geq 1$

$\Rightarrow$ $\{a_n\}$ is bounded

(c) $a_n = \frac{n!}{2^n}$

Note that $a_n = \frac{n}{2} \frac{(n-1)}{2} \cdots \frac{2}{2} \cdot \frac{1}{2} \geq \frac{n}{2} \cdot \frac{2}{2} \cdot \frac{1}{2} = \frac{n}{4}$

$\Rightarrow$ $a_n$ cannot be bounded

(d) $a_n = n^{1/n}$

Note that $\lim_{n\to\infty} n^{1/n} = \lim_{x\to\infty} x^{1/x} \underset{\uparrow \text{L'Hospital}}{=} 1 \Rightarrow$ bounded

Check! Not a monotonic sequence

(e) $a_1 = 1,\ a_n = 3 - \frac{1}{a_{n-1}},\ n \geq 2$

Note, $0 \leq a_n \leq 3 \Rightarrow$ $a_n$ is bounded.

5. Find the limit of the sequence $\{\sqrt{3}, \sqrt{3\sqrt{3}}, \sqrt{3\sqrt{3\sqrt{3}}}, \ldots\}$.

$$a_n = 3^{\sum_{k=1}^{n} \frac{1}{2^k}}$$

$$\lim_{n\to\infty} a_n = 3^{\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2^k}} = 3^{\sum_{k=1}^{\infty} \frac{1}{2^k}} = 3 \quad \underline{\underline{\text{Ans}}}$$

6. Show that the sequence $\{a_n\}$ defined by $a_1 = 1/2, a_{n+1} = \frac{1}{2-a_n}$ satisfies $0 \le a_n \le 1$ and is decreasing. Deduce that the sequence is convergent and find its limit.

Let $P(n)$ be the statement that $a_{n+1} < a_n$ and $0 \le a_n \le 1$.

1. $P(1)$: $a_2 = \frac{1}{2-\frac{1}{2}} = \frac{2}{3} < \frac{1}{2} = a_1$ and $0 \le a_1 = \frac{1}{2} < 1$

2. Assume that $a_{n+1} < a_n$ and $0 \le a_n \le 1$.

3. We show that $a_{n+2} < a_{n+1}$ and $0 \le a_{n+1} \le 1$

<u>Know</u>: $a_{n+1} < a_n$ and $0 \le a_n \le 1$

$\Leftrightarrow$ $2 - a_{n+1} > 2 - a_n$ and $2 \geq 2 - a_n \geq 2 - 1 = 1$

$\Leftrightarrow$ $\frac{1}{2-a_{n+1}} < \frac{1}{2-a_n}$ and $0 \le \frac{1}{2} \le \frac{1}{2-a_n} \le 1$

$\Leftrightarrow$ $a_{n+2} < a_{n+1}$ and $0 \le a_{n+1} \le 1$



This proves that the sequence is convergent since it is bounded and monotonically increasing. Suppose $\lim_{n\to\infty} a_n = L$, then $L = \frac{1}{2-L} \Leftrightarrow 2L - L^2 - 1 = 0 \Leftrightarrow \boxed{L = 1}$ $\underline{\underline{\text{Ans}}}$

7. Determine whether or not the following series converge?

(a) $\sum(\frac{1}{k} - \frac{1}{k!})$

Diverges by LCT with $b_K = \frac{1}{K}$

(b) $\sum \sin(\frac{\pi}{k^2})$

Converges by LCT with $b_K = \frac{1}{K^2}$

(c) $\sum(-1)^k(\sqrt{k+1} - \sqrt{k})$

Converges by AST

1. $\lim_{K \to \infty} \sqrt{K+1} - \sqrt{K} = 0$ (Rationalize)
2. $a_{K+1} < a_K$, $a_K = \sqrt{K+1} - \sqrt{K}$
3. $a_K = \sqrt{K+1} - \sqrt{K} \geq 0$.

(d) $\sum(-1)^k \frac{(k!)^2}{(2k)!}$ $\quad a_K = \frac{(K!)^2}{(2K)!} \geq 0$

$$\lim_{K \to \infty} \left| \frac{(-1)^{K+1} a_{K+1}}{(-1)^K a_K} \right| = \lim_{K \to \infty} \frac{(K+1)^2}{(2K+2)(2K+1)} = \frac{1}{4} < 1$$

Converges by Ratio test.

(e) $\sum \frac{\cos(\pi k/4)}{k^2}$

Note: $\left|\cos\left(\frac{\pi K}{4}\right)\right| \leq 1 \Rightarrow \left|\frac{\cos\left(\frac{\pi K}{4}\right)}{K^2}\right| \leq \frac{1}{K^2}$

$\Rightarrow \sum \left| \frac{\cos \frac{\pi K}{4}}{K^2} \right|$ converges by BCT with $b_K = \frac{1}{K^2}$

$\Rightarrow \sum_{K=1}^{\infty} \frac{\cos \frac{\pi K}{4}}{K^2}$ converges absolutely $\Rightarrow$ Convergent

8. If the sum of the first six terms of a geometric series is 5.25 and the common ratio is $-\frac{1}{2}$, then find the first term of the series.

$$\sum_{n=0}^{5} ar^n = 5.25 = \frac{a(1-r^6)}{1-r}$$

But $r = -\frac{1}{2}$

$$\Rightarrow \frac{(5.25)}{1-\frac{1}{64}}\left(\frac{3}{2}\right) = a \Rightarrow a = \frac{(5.25)(96)}{63} = 8.$$

9. If the fourth term in geometric series is $\frac{4}{3}$ and the seventh term is $\frac{32}{81}$, then find the value of the common ratio.

$$\boxed{r = \frac{2}{3}}$$

10. Then find the sum of the first nine terms of the geometric series that has $a_4 = 48$ and $a_6 = 192$..

$$a_4 = ar^3 = 48, \quad a_6 = ar^5 = 192$$

$$r^2 = \frac{192}{48} = 4 \Rightarrow r = \pm 2 \text{ and } a = 6 \quad (\text{Ignore } -2 \because a_4 > 0.)$$

$$S_9 = \sum_{k=0}^{8} ar^k = 6\left(\frac{1-2^9}{1-2}\right) = 3066 \text{ Ans}$$

11. In the first stage of a chain email, four people send a message to four of their friends. Then what are the number of stages(to the nearest whole number) required for one million people to have received the email?

Ist stage $\longmapsto 4$

IInd stage $\longmapsto 4(4) = 4^2$

$\vdots$

$n^{th}$ stage $\longmapsto 4^n$

This means $4 + 4^2 + 4^3 + \dots + 4^{n-1} = 10^6$

$$\Rightarrow \frac{4(1-4^n)}{1-4} = 10^6 \Rightarrow 750001 = 4^n$$

$$\Rightarrow n = \frac{\log 750001}{\log 4} = 9.75 \approx 10 \text{ Ans}$$

12. Estimate the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ with an error $< 0.005$.

This is same as finding the sum correct to two decimals.

$$a_n = \frac{1}{n^4}$$

$$\frac{1}{(n+1)^4} = a_{n+1} < 0.005 = \frac{5}{1000}$$

$$\frac{1000}{5} < (n+1)^4$$

$\underline{\underline{n = 3}}$ $\quad 200 < 256 \quad$ True

$\underline{\underline{n = 2}}$ $\quad 200 \nless 81 \quad$ False

This shows that the sum of the series $\sum \frac{(-1)^{n+1}}{n^4}$ with an error $< 0.005$ is

$$\frac{(-1)^2}{1^4} + \frac{(-1)^3}{2^4} + \frac{(-1)^4}{3^4}$$

$$= 1 - \frac{1}{16} + \frac{1}{81} = \frac{1199}{1296} \approx 0.9251 \quad \underline{\underline{\text{Ans}}}$$

13. Ex 8.2 Problem 4

$1+0.4+0.16+0.064+\ldots$ is a geometric series with Common ratio $= 0.4 = \frac{2}{5} < 1 \implies$ The given series Converges to $\frac{a}{1-r} = \frac{1}{1-\frac{2}{5}} = \frac{5}{3}$ Ans

14. Ex 8.2 Problem 12

diverges by divergence test since $\lim_{k\to\infty} \frac{k(k+2)}{(k+3)^2} = 1 \neq 0$

15. Ex 8.2 Problem 14

$a_n = \frac{1+3^n}{2^n} > \frac{3^n}{2^n} = b_n$. $\sum b_n$ diverges since $r = \frac{3}{2} > 1$

Thus, by comparison test $\sum_{n=1}^{\infty} a_n$ diverges.

16. Ex 8.2 Problem 17

$$\sum_{n=1}^{\infty} \left[(0.8)^{n-1} - (0.3)^n\right] \underset{\text{b/c each is convergent}}{\underline{\underline{\text{split}}}} \sum_{n=1}^{\infty} (0.8)^{n-1} - \sum_{n=1}^{\infty} (0.3)^n = \frac{1}{0.2} - \frac{0.3}{0.7}$$

converges by GST

Ans

17. Ex 8.2 Problem 18

$\sum_{k=1}^{\infty} (\cos 1)^k$ is a geometric series with common ratio $r = \cos 1 < 1$ and $\sum (\cos 1)^k = \frac{\cos 1}{1-\cos 1}$ Ans

18. Ex 8.2 Problem 47(b)

Area removed at the first step is $\frac{1}{9}$;

second step is $8\left(\frac{1}{9}\right)^2$;

$\vdots$

$n^{th}$ step is $8^{n-1}\left(\frac{1}{9}\right)^n$

So, the total area of all removed square is

$$\sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{8}{9}\right)^{n-1} = \frac{\frac{1}{9}}{1-\frac{8}{9}} = 1 \quad \underline{\underline{\text{Ans}}}$$

19. Ex 8.2 Problem 50 (Not for exam... but interesting problem to think about.)

$$\text{Area} = \frac{11\pi}{96}$$

Circles occupy about 83.1% of the area of the triangle

20. Ex 8.3 Problem 23

Note that $\frac{2+(-1)^n}{n\sqrt{n}} < \frac{3}{n\sqrt{n}}$

$\sum \frac{3}{n\sqrt{n}}$ converges by p-series test since $p = \frac{3}{2} > 1$

$\sum \frac{2+(-1)^n}{n\sqrt{n}}$ converges by comparison test.

21. Ex 8.3 Problem 24

Note: $\frac{1+\sin n}{10^n} \geq 0$ and $\frac{1+\sin n}{10^n} \leq \frac{2}{10^n}$

$\sum \frac{2}{10^n}$ converges since it is a geometric series with common ratio $= \frac{1}{10} < 1$

$\Rightarrow \sum \frac{1+\sin n}{10^n}$ converges by comparison test.

22. Ex 8.3 Problem 25

$a_n = \sin\left(\frac{1}{n}\right)$

Note: $a_n \geq 0$ and $\lim_{n\to\infty} \frac{a_n}{\frac{1}{n}} = 1 > 0$ and finite

$\Rightarrow \sum a_n$ diverges by LCT since $\sum \frac{1}{n}$ diverges.

23. Ex 8.3 Problem 27

Take the related function $f(x) = \frac{1}{x(\ln x)^p}$

1. $f(x)$ is clearly continuous for all $x > 1$.

2. $f(x) \geq 0$ for all $x > 1$

3. $f'(x) = \frac{-p + \ln x}{x^2(\ln x)^{p+1}} < 0$ if $x > e^{-p}$

which means that $f$ is eventually decreasing.

This allows us to apply Integral test:

$$\int_2^\infty \frac{1}{x(\ln x)^p}\, dx = \begin{cases} \lim_{t\to\infty} \frac{(\ln t)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p} & p \neq 1 \\ \lim_{t\to\infty} \ln(\ln t) - \ln(\ln 2) & \end{cases}$$

$$= \begin{cases} -\frac{(\ln 2)^{1-p}}{1-p} & p > 1 \\ \text{d.n.e} & p < 1 \\ \text{d.n.e} & p \neq 1 \end{cases}$$

This shows that $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges for all $p > 1$.

24. Ex 8.4 Problem 8 $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln n}{n} = \sum_{n=2}^{\infty} (-1)^{n-1} \frac{\ln n}{n}$

Note: $b_n = \frac{\ln n}{n} \geq 0$ for $n \geq 2$ and $\lim_{n\to\infty} b_n = 0$

$f(x) = \frac{\ln x}{x}$, $f'(x) = \frac{1-\ln x}{x^2} < 0$ for $x > e$ which means that $\{b_n\}$ is eventually decreasing.

So the series converges by AST. This is an example of conditionally convergent sequence

25. Ex 8.4 Problem 24

$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^n}{n^4}$ diverges by divergence test.

Note: $\lim_{n\to\infty} \frac{2^n}{n^4} = \infty$ so, $\lim_{n\to\infty} \frac{2^n}{n^4}(-1)^n$ does not exist.

26. Ex 8.4 Problem 27

Note $\left|\frac{\cos\left(\frac{n\pi}{3}\right)}{n!}\right| < \frac{1}{n!}$ and $\sum \frac{1}{n!}$ converges by ratio test.

So, $\sum \frac{\cos \frac{n\pi}{3}}{n!}$ converges absolutely by comparison test.

27. Ex 8.4 Problem 42

(a) $a_n = \frac{2\sqrt{2}}{9801} \frac{(4n)!\ (1103 + 26390n)}{(n!)^4\ (396)^{4n}}$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{99^4} < 1$, so by the Ratio test the series $\sum a_n$ converges.

(b) $\frac{1}{\pi} \approx \frac{2\sqrt{2}}{9801} \frac{1103}{(396)^{4(0)}} = \frac{2\sqrt{2}}{9801}(1103) \Rightarrow \pi \approx 3.14159273$

Correct to 6 decimal places

With $n=1$ $\frac{1}{\pi} \approx \frac{2\sqrt{2}}{9801}\left(\frac{1103}{1} + \frac{4!\,(1103+26390)}{396^4}\right)$

$\Rightarrow \pi \approx 3.141592653589793878 \longrightarrow$ correct to 15 decimals.

28. Mark True or False.

(a) Every unbounded sequence is divergent. True

(b) If $\lim_{n\to\infty} a_n = 0$ then $\sum a_n$ converges. False; eg $a_n = \frac{1}{n}$

(c) If $a_n$ converges then $\lim_{n\to\infty} a_n = 0$ True

(d) Every telescoping series converges. False; eg $a_n = \ln\left(\frac{n}{n+1}\right)$

(e) Every alternating series converges. False; eg. $a_n = \sum_{k=1}^{\infty} (-1)^k$

(f) Every absolutely convergent series is convergent. True

(g) Every convergent series is absolutely convergent. False; $\sum \frac{(-1)^k}{k}$

(h) If $\sum |a_n|$ is divergent then $\sum a_n$ is also divergent. True

(i) If $\sum {a_n}^2$ is convergent then $\sum a_n$ is absolutely convergent. False; $a_n = \frac{1}{n}$

(j) If $\sum {a_n}^2$ is convergent then $\sum a_n$ is convergent. False; $a_n = \frac{1}{n}$

(k) If $\sum a_n$ is convergent then $\sum {a_n}^2$ is convergent. True

(l) If $0 \le a_n \le b_n$ for every $n$ and $\sum b_n$ diverges then $\sum a_n$ converges. False; $a_n = \frac{1}{n+1} < \frac{1}{n} = b_n$

(m) The ratio test can be used to determine whether $\sum \frac{1}{n^2}$. False; $\frac{a_{n+1}}{a_n} \to 1$ Ratio test fails

(n) Let $a_n$ be a given series. If the sequence of partial sum is bounded then the series $\sum a_n$ converges. False; eg: $\sum(-1)^k$

(o) Let $a_n$ be a given series of positive terms. If the sequence of partial sum is bounded then the series $\sum a_n$ converges. True

(p) If $\{a_k\}$ is a decreasing sequence of positive numbers that converge to 0 then the alternating series $\sum(-1)^k a_k$ necessarily converge. True