1. Find $f(t)$ given that $f'(t) = \sin t\,\mathbf{i} + 3t^2\,\mathbf{j}$ and $f(0) = \mathbf{i} - \mathbf{k}$.

$$\vec{f}(t) = -\cos t\,\hat{i} + t^3\,\hat{j} + \vec{C}$$

Note: $\vec{f}(0) = -\hat{i} + \vec{C} \underset{\text{given}}{=} \hat{i} - \hat{k} \Rightarrow \vec{C} = 2\hat{i} - \hat{k}$

$$\vec{f}(t) = -\cos t\,\hat{i} + t^3\,\hat{j} + 2\hat{i} - \hat{k}$$

$$\boxed{\vec{f}(t) = (2 - \cos t)\,\hat{i} + t^3\,\hat{j} - \hat{k}}$$ Ans

2. Find $\lim_{t\to 0} 3(t^2 - 1)\,\mathbf{i} + \cos t\,\mathbf{j} + \frac{t}{|t|}\,\mathbf{k}$.

$\downarrow$ $-3$ $\quad \downarrow$ $1$ $\quad \downarrow$ d.n.e

This shows that $\boxed{\lim_{t\to 0} 3(t^2-1)\,\hat{i} + \cos t\,\hat{j} + \frac{t}{|t|}\,\hat{k} \text{ d.n.e.}}$ Ans

3. Find the points on the curve $\mathbf{r}(t)$ at which $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ have opposite directions.

$$\mathbf{r}(t) = 5t\,\mathbf{i} + (3 + t^2)\,\mathbf{j}$$

$\vec{r}\,'(t) = 5\hat{i} + 2t\,\hat{j}$ and $\vec{r}(t)$ have opposite direction

$\Rightarrow \vec{r}\,'(t) = \alpha\,\vec{r}(t), \quad \alpha < 0.$

$\Rightarrow \langle 5, 2t \rangle = \alpha \langle 5t, 3 + t^2 \rangle$

$\Rightarrow 5 = 5\alpha t$ and $2t = (3 + t^2)\alpha$

$\Rightarrow \alpha = \frac{1}{t} \xrightarrow{\text{Plug in}} 2t = (3 + t^2)\frac{1}{t} = \frac{3}{t} + t$

$t \neq 0 \because \alpha t = 1$ $\quad \Rightarrow t = \frac{3}{t} \Rightarrow t^2 = 3$

$\Rightarrow t = \pm\sqrt{3}$

$\Rightarrow \boxed{t = -\sqrt{3}}$ $\quad (\because t < 0$ b/c $\alpha < 0)$

Point $(-5\sqrt{3}, 6)$ Ans

4. Find a vector function that represents the curve of intersection of the two surfaces.

(a) The cylinder $x^2 + y^2 = 4$ and the surface $z = xy$.

The projection of the curve $C$ of intersection onto the xy-plane is the circle $x^2 + y^2 = 4$, $z = 0$.

Then we can write $x = 2\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$.

Since $C$ also lies on the surface $z = xy$, we have $z = xy = (2\cos t)(2\sin t) = 4\cos t \sin t$

The parametric equations for $C$ are:

$$x = 2\cos t, \quad y = 2\sin t, \quad z = 4\cos t \sin t, \quad 0 \le t \le 2\pi,$$

and the corresponding vector function is

$$\boxed{\vec{r}(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j} + 2\sin(2t)\,\hat{k}, \quad 0 \le t \le 2\pi}$$

(b) The cone $z = \sqrt{x^2 + y^2}$ and the plane $z = x + 1$ ⟵ Fix the typo.

Both equations are solved for $z$, so we can substitute to eliminate $z$: $\sqrt{x^2 + y^2} = x + 1$

$$\Rightarrow \quad x^2 + y^2 = (x+1)^2$$

$$\Rightarrow \quad x^2 + y^2 = x^2 + 1 + 2x$$

$$\Rightarrow \quad \boxed{y^2 = 1 + 2x}$$

We can form a parametric equation of curve $C$ by choosing $y = t$, then $x = \frac{t^2 - 1}{2}$ and $z = 1 + \frac{t^2 - 1}{2}$

$$\Rightarrow \quad \boxed{C: \ r(t) = \frac{t^2 - 1}{2}\,\hat{i} + t\,\hat{j} + \frac{t^2 + 1}{2}\,\hat{k}}$$

5. Find the point at which the following curves intersect. Also the find angle of intersection.

$$\mathbf{r}_1(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}, \quad \mathbf{r}_2(u) = (1+u)\,\mathbf{i} + u^2\,\mathbf{j} + \frac{1}{8}\,\mathbf{k}$$

Intersect: $t = 1 + u$ — 1

$t^2 = u^2$ — 2

$t^3 = \frac{1}{8} \Rightarrow \boxed{t = \frac{1}{2}} \xRightarrow[1]{\text{Using}} \boxed{u = -\frac{1}{2}}$

Check the answer by plugging in eqn 2

$\left(\frac{1}{2}\right)^2 = \left(-\frac{1}{2}\right)^2$ ✓ True

Point : $\boxed{\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}\right)}$ Ans.

Angle: $\cos\theta = \dfrac{\vec{r}_1{}'\left(\frac{1}{2}\right) \cdot \vec{r}_2{}'\left(-\frac{1}{2}\right)}{\left|\vec{r}_1{}'\left(\frac{1}{2}\right)\right| \left|\vec{r}_2{}'\left(-\frac{1}{2}\right)\right|} = \dfrac{\langle 1, 2\left(\frac{1}{2}\right), 3\left(\frac{1}{2}\right)^2\rangle \cdot \langle 1, 2\left(-\frac{1}{2}\right), 0\rangle}{\sqrt{1+1+\frac{3}{4}}\ \sqrt{1+1+0}}$

$= 0 \Rightarrow \boxed{\theta = \frac{\pi}{2}}$ Ans

6. A particle moves so that $\mathbf{r}(t) = 2\,\mathbf{i} + t^2\,\mathbf{j} + (t-1)^2\,\mathbf{k}$. At what time is the speed a minimum?

$\vec{r}\,'(t) = \langle 0, 2t, 2(t-1)\rangle$

Speed $= |\vec{r}\,'(t)| = \sqrt{4t^2 + 4(t-1)^2}$

Minimemizing speed is same as minimizing

$4t^2 + 4(t-1)^2 = (\text{speed})^2$

$f(t) = 4t^2 + 4(t-1)^2$

$f'(t) = 0 \Rightarrow 8t + 8(t-1) = 0$

$\Rightarrow 16t = 8 \Rightarrow \boxed{t = \frac{1}{2}}$ Ans

Minimum speed $= \sqrt{4\frac{1}{4} + 4\frac{1}{4}} = \sqrt{2}$

7. Find the arc length of the curve $2\,\mathbf{i} + t^2\,\mathbf{j} + (t-1)^2\,\mathbf{k}$. from $t = 0$ to $t = 1$.

Let $C: \vec{r}(t) = 2\hat{i} + t^2\hat{j} + (t-1)^2\hat{k}$

$$L(C) = \int_0^1 |\vec{r}\,'(t)|\,dt$$

$$= \int_0^1 \sqrt{4t^2 + 4(t-1)^2}\,dt$$

$$= 2\int_0^1 \sqrt{t^2 + (t-1)^2}\,dt$$

$$= 2\sqrt{2}\int_0^1 \sqrt{\left(t-\frac{1}{2}\right)^2 + \frac{1}{4}}\,dt$$

Set $t - \frac{1}{2} = \frac{1}{2}\tan\theta$ when $t = 0$ then $\tan\theta = -1$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ $\Rightarrow \theta = -\frac{\pi}{4}$

$dt = \frac{1}{2}\sec^2\theta\,d\theta$ and $t = 1$ then $\tan\theta = 1$ $\Rightarrow \theta = \frac{\pi}{4}$.

$$L(C) = 2\sqrt{2}\int_{-\frac{\pi}{4}}^{\pi/4} \sqrt{\frac{1}{4}\tan^2\theta + \frac{1}{4}}\;\frac{1}{2}\sec^2\theta\,d\theta$$

$$= 2\sqrt{2}\int_{-\pi/4}^{\pi/4} \frac{1}{2}\sec\theta \cdot \frac{1}{2}\sec^2\theta\,d\theta$$

$$= \frac{2\sqrt{2}}{4}\int_{-\pi/4}^{\pi/4} \sec^3\theta\,d\theta$$

I.P
$$= \frac{\sqrt{2}}{2}\left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|\right]_{-\pi/4}^{\pi/4}$$

$$= \frac{\sqrt{2}}{4}\left[\sqrt{2} + \ln|\sqrt{2}+1| - \sqrt{2}(-1) - \ln|\sqrt{2}-1|\right]$$

$$\boxed{L(C) = \frac{\sqrt{2}}{4}\left[2\sqrt{2} + \ln\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|\right]}$$ Ans

8. Find the position, velocity, acceleration vector at $t = 0$ of the particle moving along the curve $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 2\sqrt{(2)}t\mathbf{k}$. Also, find the speed and the curvature of this curve at the same point. After finding the curvature, reparametrize the given curve in terms of arc length starting at $t=0$.

Sol$^n$ Position at $t=0$: $\vec{r}(0) = \langle 2, 0, 0 \rangle$

Velocity at $t=0$: $\vec{r}\,'(0) = \langle 0, 2, 2\sqrt{2} \rangle$

Acceleration at $t=0$: $\vec{r}\,''(0) = \langle -2, 0, 0 \rangle$

Speed at $t=0$: $|\vec{r}\,'(0)| = \sqrt{4+8} = \sqrt{12}$

Curvature at $t=0$: $\dfrac{|T'(0)|}{|\vec{r}\,'(0)|} = \kappa(0)$

$$T(t) = \frac{r'(t)}{|\vec{r}\,'(t)|} = \frac{\langle -2\sin t, 2\cos t, 2\sqrt{2} \rangle}{\sqrt{4+8}}$$

$$T'(t) = \frac{\langle -2\cos t, -2\sin t, 0 \rangle}{\sqrt{12}}$$

$$T'(0) = \frac{\langle -2, 0, 0 \rangle}{\sqrt{12}} \Rightarrow \kappa(0) = \frac{2/\sqrt{12}}{\sqrt{12}}$$

$$\boxed{\kappa(0) = \frac{1}{6}}$$

Arc length function $s(t) = \int_0^t |\vec{r}\,'(u)|\,du$

$$= \int_0^t \sqrt{4\sin^2 u + 4\cos^2 u + 8}\,du$$

$$= \int_0^t \sqrt{12}\,du = \sqrt{12}\,t$$

Reparametrize: $\boxed{\vec{r}(s) = 2\cos\frac{s}{\sqrt{12}}\,\hat{i} + 2\sin\frac{s}{\sqrt{12}}\,\hat{j} + 2\sqrt{2}\,\frac{s}{\sqrt{12}}\,\hat{k}}$ Ans

9. Interpret $\mathbf{r}(t)$ as the position of a moving object at time t. Determine the normal and tangential components of acceleration at time $t = 0$.

$$\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 2\sqrt{2}t\mathbf{k}$$

Normal Components of acceleration at $t = 0$

$$= a_N(0) = \frac{|\vec{r}\,'(0) \times \vec{r}\,''(0)|}{|\vec{r}\,'(0)|} = \frac{|\langle 0, 2, 2\sqrt{2}\rangle \times \langle -2, 0, 0\rangle|}{\sqrt{4+8}}$$

$$= \frac{|4\hat{k} - 4\sqrt{2}\hat{j}|}{\sqrt{12}} = \frac{\sqrt{16+32}}{\sqrt{12}}$$

$$= \boxed{2} \text{ Ans.}$$

Tangential Components of acceleration at $t = 0$

$$= a_T(0) = \frac{\vec{r}\,'(0) \cdot \vec{r}\,''(0)}{|\vec{r}\,'(0)|} = \frac{\langle 0, 2, 2\sqrt{2}\rangle \cdot \langle -2, 0, 0\rangle}{\sqrt{12}}$$

$$= \boxed{0} \text{ Ans}$$

10. Find the velocity and position vector given that $\mathbf{a}(t) = \mathbf{i} + 2\mathbf{j}$, $v(0) = \mathbf{k}$, and $\mathbf{r}(0) = \mathbf{i}$.

Velocity vector: $\vec{v}(t) = \int_0^t \vec{a}(u)\,du$

Position vector: $\vec{r}(t) = \int_0^t \vec{v}(u)\,du$

$$\vec{v}(t) = \int_0^t (\hat{i} + 2\hat{j})\,du$$

$$= t\hat{i} + 2t\hat{j} + \vec{C}$$

$$\vec{v}(0) = \vec{C} = \hat{k}$$

$$\Rightarrow \boxed{\vec{v}(t) = t\hat{i} + 2t\hat{j} + \hat{k}}$$

Similarly: $\vec{r}(t) = \frac{t^2}{2}\hat{i} + t^2\hat{j} + t\hat{k} + \vec{D}$

$$\vec{r}(0) = \vec{D} = \hat{i}$$

$$\Rightarrow \boxed{\vec{r}(t) = \left(\frac{t^2}{2} + 1\right)\hat{i} + t^2\hat{j} + t\hat{k}}$$

11. Find the radius of curvature of the curve $2y = x^2$ at $x = 0$.

$$y = \frac{x^2}{2} \Rightarrow y' = x \text{ and } y'' = 1$$

$$\kappa(x) = \frac{|y''|}{(1+(y')^2)^{3/2}} = \frac{1}{(1+x^2)^{3/2}}$$

$$\kappa(0) = \frac{1}{(1+0^2)^{3/2}} = 1$$

Radius of curvature at $x = 0$ is given by: $\boxed{R(0) = \frac{1}{\kappa(0)} = 1}$ Ans

12. Find the curvature of the curve $\mathbf{r}(t) = 2t\,\mathbf{i} + t^3\,\mathbf{j}$ in terms of $t$.

$x(t) = 2t$

$y(t) = t^3$

$$\kappa(t) = \frac{|x'(t)y''(t) - y'(t)x''(t)|}{[x'^2 + y'^2]^{3/2}}$$

$$= \frac{|2(6t) - (3t^2)(0)|}{(4 + 9t^4)^{3/2}}$$

$$\boxed{\kappa(t) = \frac{12|t|}{(4+9t^4)^{3/2}}}$$ Ans

13. Find the domain and range of the function $f(x, y, z) = \frac{z^2}{x^2-y^2}$.

$$\text{Domain} = \{(x, y, z) : x^2 - y^2 \neq 0\}$$

In other words, domain is the set of all those points in the space except for the points which lie on the plane $x - y = 0$ or $x + y = 0$

$$\text{Range} = (-\infty, \infty).$$

14. Find the domain and range of the function $f(x, y, z) = \frac{x+y+z}{|x+y+z|}$.

$$\text{Domain} = \{(x, y, z) : x + y + z \neq 0\}$$

$$\text{Range} = \{1, -1\}$$

15. List equations of all 6 types of the quadric surfaces.

1. Ellipsoid $\longrightarrow$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
2. Elliptic paraboloid $\longrightarrow$ $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
3. Hyperbolic paraboloid $\longrightarrow$ $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$
4. Cone $\longrightarrow$ $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$
5. Hyperboloid of one sheet $\longrightarrow$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
6. Hyperboloid of two sheets $\longrightarrow$ $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

16. Identify the surface $9x^2 + 4y^2 - 36z^2 + 36 = 0$ and find its traces.

Hyperboloid of two sheets : $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{1} = -1$

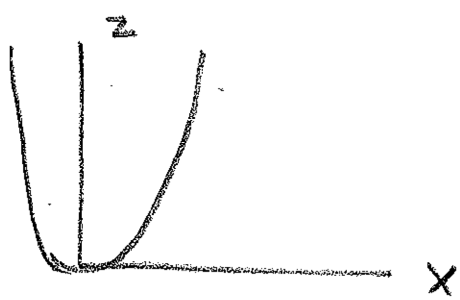
Hyperboloid two sheets

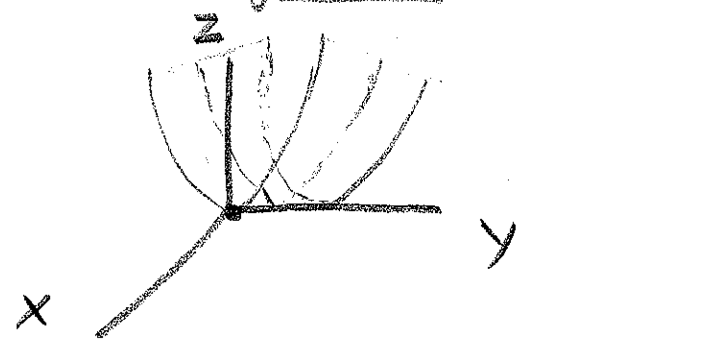
$\underline{\text{xy-trace}}$ : $9x^2 + 4y^2 + 36 = 0$ $\underline{\text{Ellipse}}$

$\underline{\text{yz-trace}}$ : $4y^2 - 36z^2 + 36 = 0$ $\underline{\text{Hyperbola}}$

$\underline{\text{zx-trace}}$ : $9x^2 - 36z^2 + 36 = 0$ $\underline{\text{Hyperbola}}$

17. Sketch the cylinder $z = x^2$.




18. The surfaces $x^2 + y^2 + (z-2)^2 = 2$ and $x^2 + y^2 = z^2$ intersect in a space curve C. Determine the projection of C onto the xy-plane.

$z^2 + (z-2)^2 = 2$ "Eliminate x and y"

$z^2 + z^2 + 4 - 4z = 2$

$\Rightarrow \boxed{2z^2 - 4z + 4 = 2} \Rightarrow z^2 - 2z + 1 = 0$

$\Rightarrow (z-1)^2 = 0 \Rightarrow \boxed{z = 1}$

Plug it in any the eqn:

C : $\boxed{x^2 + y^2 = 1}$ Circle!

19. Identify the level curves $f(x,y) = c$ and sketch the curves corresponding to the indicated values of c.

(a) $f(x,y) = e^{xy}$ where $c = 0, 1, 2$.

$e^{xy} = c$

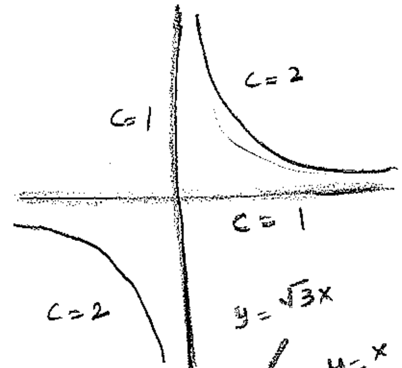
$\Leftrightarrow xy = \ln c$

$c = 0$: No level curve

$c = 1$: $xy = \ln 1 = 0$

$\Leftrightarrow x = 0$ or $y = 0$

$c = 2$: $xy = \ln 2$



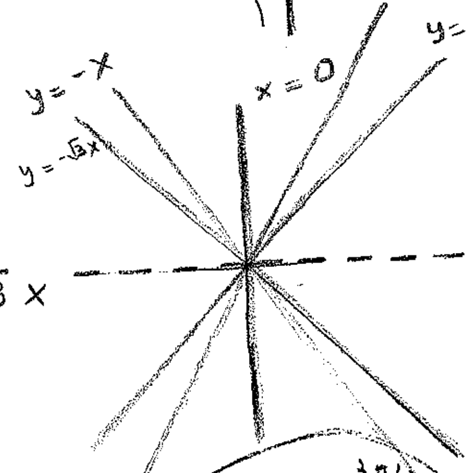

(b) $f(x,y) = \frac{x^2}{x^2+y^2}$ where $c = 0, 1/4, 1/2$.

$\frac{x^2}{x^2+y^2} = c$

$c = 0$: $\frac{x^2}{x^2+y^2} = 0 \Rightarrow x = 0$

$c = \frac{1}{4}$: $\frac{x^2}{x^2+y^2} = \frac{1}{4} \Leftrightarrow y^2 = 3x^2 \Leftrightarrow y = \pm\sqrt{3}\,x$

$c = \frac{1}{2}$: $\frac{x^2}{x^2+y^2} = \frac{1}{2} \Leftrightarrow y^2 = x^2 \Leftrightarrow y = \pm x$

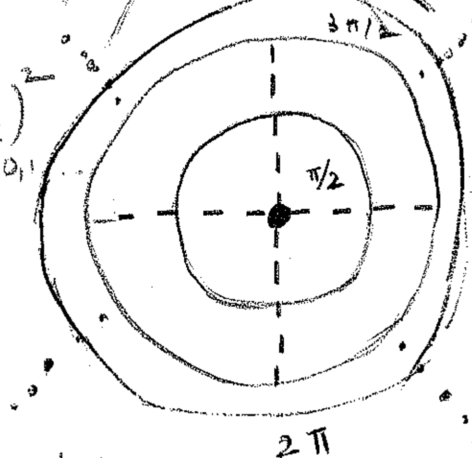



(c) $f(x,y) = \cos\sqrt{x^2+y^2}$ where $c = 0, 1$.

$\cos\sqrt{x^2+y^2} = c$

$c = 0$: $\sqrt{x^2+y^2} = (2n+1)\frac{\pi}{2} \Rightarrow x^2+y^2 = \left((2n+1)\frac{\pi}{2}\right)^2$ for all $n = 0, 1, \ldots$

$c = 1$: $\sqrt{x^2+y^2} = 2n\pi \Rightarrow x^2+y^2 = (2n\pi)^2$ for all $n = 0, 1, 2, \ldots$

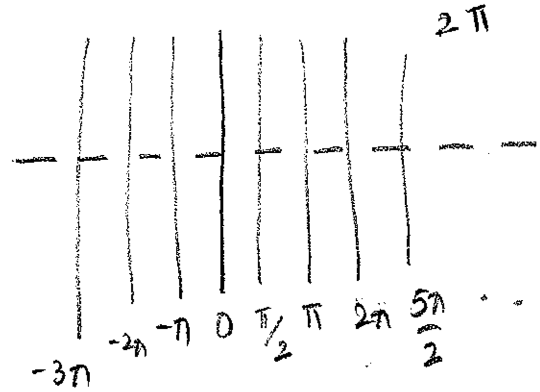



(d) $f(x,y) = \sin x$ where $c = 0, 1$.

$\sin x = c$

$c = 0$: $x = 0, \pi, 2\pi, 3\pi, \ldots, -\pi, -2\pi, -3\pi, \ldots$

$c = 1$: $x = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$



20. Identify the level surfaces $f(x, y, z) = c$.

(a) $f(x, y) = x + y + 3z$ where $c = 0$.

z

$X + y + 3z = 0 \implies$ plane!

(b) $f(x, y, z) = x^2 + y^2$ where $c = 4$.

$x^2 + y^2 = 4$ and $z$ is arbitrary

$\implies$ Infinitely long cylinder with radius $= 2$

21. Find an equation for the level curve of $f(x, y) = y^2 \arctan x$ that contains the point $P(1, 2)$.

Find the value of c: $c = f(1,2) = 4\frac{\pi}{4} = \pi$

Eqn of the level curve: $\boxed{y^2 \arctan x = \pi}$ Ans

22. Find $f_x(1, 2)$ and $f_y(1, 2)$ given that $f(x, y) = \frac{x}{x+y}$.

$$f_x(x,y) = \frac{1(x+y) - x(1)}{(x+y)^2} = \frac{y}{(x+y)^2}$$

$\boxed{f_x(1,2) = \frac{2}{9}}$ , $f_y(x,y) = -x(x+y)^{-2}$

$\boxed{f_y(1,2) = -\frac{1}{9}}$ Ans

23. The intersection of a surface $z = x^2 + y^2$ and a plane $x = 2$ is a curve C in a space. Find equation for the line tangent to C at the point P(2, 1, 5).

Eqn of the curve C: $z = 4 + y^2$, $x = 2$

Eqn of the tangent line to C at P is given by: $(z-5) = \left.\frac{\partial z}{\partial y}\right|_P (y-1)$ and $x = 2$

$\Leftrightarrow$ $\boxed{z - 5 = 2(y-1) \text{ and } x = 2}$ Ans

24. Show that the function $g(x,y) = \frac{x^2y}{x^4+y^2}$ has limiting value 0 as $(x,y) \to (0,0)$ along any line through the origin, but $\lim_{(x,y)\to(0,0)} g(x,y)$ still does not exist.

Take $y = mx$ – line through origin.

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} g(x,y) = \lim_{x\to 0} \frac{x^2(mx)}{x^4+(mx)^2} = \lim_{x\to 0} \frac{mx}{x^2+m^2} = 0$$

for all m.

Look

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x^2}} g(x,y) = \lim_{x\to 0} \frac{x^2(x^2)}{x^4+(x^2)^2} = \frac{1}{2}$$

This shows that $\lim_{(x,y)\to(0,0)} g(x,y)$ d.n.e.

25. Find $\lim_{(x,y)\to(1,1)} \frac{x^2y}{x^4+y^2}$ if it exists.

Note $(1,1)$ belongs to the domain of $\frac{x^2y}{x^4+y^2}$ and is continuous at this point.

This shows that $\lim_{(x,y)\to(1,1)} \frac{x^2y}{x^4+y^2} = \frac{1^2(1)}{1^4+1^2} = \boxed{\frac{1}{2}}$ Ans

26. Given

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & \text{when (x, y)} \neq (0,0), \\ 0 & \text{when (x, y)} = (0,0) \end{cases}$$

(a) Find $\lim_{(x,y)\to(0,0)} f(x,y)$ if it exists.

$$\lim_{\substack{(x,y)\to(0,0) \\ x=0}} f(x,y) = \lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{-y^2}{y^2} = -1$$

$$\lim_{\substack{(x,y)\to(0,0) \\ y=0}} f(x,y) = \lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

Not equal

This shows that limit d.n.e.

(b) Is this function continuous everywhere? Give proper reasoning in support of your answer.

Clearly, $f(x,y)$ is continuous at all points except at $(0,0)$ since $\lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0)$ and

for points other than $(0,0)$, it is continuous b/c it is a rational function whose domain is everything except $(0,0)$.

27. Given

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^4+y^4} & \text{when (x, y)} \neq (0,0), \\ 0 & \text{when (x, y)} = (0,0) \end{cases}$$

(a) Find $f_x(0,0)$ and $f_y(0,0)$ if they exist.

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

Similarly, $\boxed{f_y(0,0) = 0}$

(b) Find $\lim_{(x,y)\to(0,0)} f(x,y)$ if it exists.

$$\lim_{\substack{(x,y)\to(0,0)\\ y=x}} f(x,y) = \lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{x^4}{x^4+x^4} = \frac{1}{2}$$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=2x}} f(x,y) = \lim_{x\to 0} f(x,2x) = \lim_{x\to 0} \frac{x^2(2x)^2}{x^4+(2x)^4} = \frac{2^2}{1+2^4}$$

Not equal

$\Rightarrow$ Limit d.n.e.

28. Evaluate $\lim_{(x,y)\to(0,0)} \frac{y^2}{x^2+y^2}$ along:

(a) the line $y = mx$.

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{y^2}{x^2+y^2} = \lim_{x\to 0} \frac{m^2x^2}{x^2+m^2x^2} = \boxed{\frac{m^2}{1+m^2}}$$ Ans

(b) the path $\mathbf{r}(t) = \frac{1}{t}\mathbf{i} + \frac{\sin t}{t}\mathbf{j},\ t > 0$.

$x(t) = \frac{1}{t}$ and $y(t) = \frac{\sin t}{t}$

Note $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} y(t) = 0$ (by Squeeze thm)

$$\Rightarrow \lim_{\substack{(x,y)\to(0,0)\\ r(t)}} f(x,y) = \lim_{t\to\infty} f\left(\frac{1}{t}, \frac{\sin t}{t}\right) = \lim_{t\to\infty} \frac{\frac{\sin^2 t}{t^2}}{\frac{1}{t^2} + \frac{\sin^2 t}{t^2}} = \lim_{t\to\infty} \frac{\sin^2 t}{1+\sin^2 t}$$

$= \underline{\text{d.n.e}}$

Note: Existence of partial derivatives alone do not guarantee that the function is differentiable!

29. Given

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{when (x, y)} \neq (0,0), \\ 0 & \text{when (x, y)} = (0,0) \end{cases}$$

Find $f_x(0,0)$ and $f_y(0,0)$. Is this function differentiable at (0, 0)? Is this function continuous at (0, 0)?

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

Similarly, $f_y(0,0) = 0$

Note that $\lim_{(x,y)\to(0,0)} f(x,y)$ d.n.e $\Rightarrow$ function is not even continuous at $(0,0) \Rightarrow$ It is not differentiable at $(0,0)$.

30. Given

$$f(x,y) = \begin{cases} \frac{x^2y^3}{2x^2+y^2} & \text{when (x, y)} \neq (0,0), \\ 0 & \text{when (x, y)} = (0,0) \end{cases}$$

(a) Find $\lim_{(x,y)\to(0,0)} f(x,y)$ if it exists.

Note: $0 < \left|\frac{x^2y^3}{2x^2+y^2}\right| < |y|^3 \quad (\because 2x^2+y^2 > x^2)$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \left|\frac{x^2y^3}{2x^2+y^2}\right| = 0 \Rightarrow \boxed{\lim_{(x,y)\to(0,0)} f(x,y) = 0}$$

(b) Is this function continuous everywhere? Give proper reasoning in support of your answer.

Clearly, $\frac{x^2y^3}{2x^2+y^2}$ continuous everywhere except at $(0,0)$ which does not belong to the domain of $\frac{x^2y^3}{2x^2+y^2}$

Part (a) shows that $f$ is continuous at $(0,0)$

$\Rightarrow$ The given function $f$ is continuous everywhere.

31. Let $f$ be a function of $x$ and $y$ with everywhere continuous second partials. Is it possible that
$$\frac{\partial f}{\partial x} = x + y \quad \text{and} \quad \frac{\partial f}{\partial y} = y - x?$$

By Clairaut's theorem, $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2}$ (Must be true!)

Let us check: $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(x+y) = 1$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(y-x) = -1$

not equal

This shows that it is not possible to have a function with $\frac{\partial f}{\partial x} = x + y$ and $\frac{\partial f}{\partial y} = y - x$.

32. The intersection of a surface $z = x^2 + y^2 + xy$ with a plane $y = 2$ is a curve in space. Find the equation of the tangent line to the curve C at the point $P(1, 2, 7)$.

This problem is similar to problem 23.

C: $z = x^2 + 2^2 + 2x$ and $y = 2$

Eqn: $z - 7 = \left.\frac{\partial z}{\partial x}\right|_P (x-1)$ and $y = 2$

$z - 7 = (2(1) + 2)(x-1)$ and $y = 2$

$$\boxed{\begin{array}{l} z = 4(x-1) + 7 \\ y = 2 \end{array}}$$ Ans

33. Find the gradient of the function $f(x, y) = 2x + x^2 + 2y + y^2$ by using definition directly.

$$\nabla f = \langle f_x , f_y \rangle$$

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} = \lim_{h\to 0} \frac{2(x+h) + (x+h)^2 + \cancel{2y} + \cancel{y^2} - (2x + x^2 + \cancel{2y} + \cancel{y^2})}{h}$$

$$\Rightarrow \quad f_x(x,y) = \lim_{h\to 0} \frac{\cancel{2x} + 2\cancel{h} + \cancel{x^2} + h^{\cancel{2}} + 2x\cancel{h} - \cancel{2x} - \cancel{x^2}}{\cancel{h}}$$

$$= \lim_{h\to 0} 2 + h + 2x = 2 + 2x$$

similarly, $f_y(x,y) = 2 + 2y$

$$\therefore \quad \nabla f = \langle 2 + 2x , 2 + 2y \rangle$$

34. Find the gradient vector at the point P(2, 1) for the function $f(x, y) = ln(x^2 + y^2)$.

$$\nabla f(2,1) = \langle f_x(2,1) , f_y(2,1) \rangle$$

$$= \left\langle \frac{2(2)}{2^2 + 1^2} , \frac{2(1)}{2^2 + 1^2} \right\rangle$$

$$= \left\langle \frac{4}{5} , \frac{2}{5} \right\rangle \quad \underline{\underline{\text{Ans}}}$$

35. Find $\nabla[f(0,0)g(0,0)]$ given that $\nabla f(0,0) = \mathbf{i} - \mathbf{j}$, $\nabla g(0,0) = \mathbf{i} + \mathbf{j}$ and $f(0,0) = 1$, $g(0,0) = 2$.

$$\nabla\left[f(0,0)\cdot g(0,0)\right] \overset{\text{product rule}}{\underset{\text{verify it}}{=}} \nabla f(0,0)\, g(0,0) + f(0,0)\,\nabla g(0,0)$$

$$= 2(\hat{i}-\hat{j}) + 1(\hat{i}+\hat{j}) = \boxed{3\hat{i} - \hat{j}} \text{ Ans}$$

Verification: $\nabla(f(x,y)g(x,y)) = \langle (fg)_x, (fg)_y \rangle$

Note that product rule of partial derivative is same as product rule of regular derivative

$$= \langle f_x g + f g_x,\ f_y g + f g_y \rangle$$

$$= \langle f_x g, f_y g \rangle + \langle f g_x, f g_y \rangle$$

$$= \langle f_x, f_y \rangle g + f\langle g_x, g_y \rangle$$

$$= \nabla f\, g + f\, \nabla g$$

36. Find the directional derivative of the function $f(x,y) = 2x^2 + 3y$ at the point P(1, 1) in the direction of the vector $2\mathbf{i} - 3\mathbf{j}$.

Find $D_{\hat{u}} f(1,1)$ where $\hat{u} = \dfrac{2\hat{i} - 3\hat{j}}{|2\hat{i} - 3\hat{j}|} = \dfrac{\langle 2, -3\rangle}{\sqrt{13}}$

$$D_{\hat{u}} f(1,1) = \nabla f(1,1)\cdot \hat{u}$$

$$= \langle 4x, 3\rangle\big|_P \cdot \hat{u}$$

$$= \langle 4, 3\rangle \frac{\langle 2, -3\rangle}{\sqrt{13}}$$

$$= -\frac{1}{\sqrt{13}} \text{ Ans}$$

37. Determine the minimum directional derivative of $f(x, y) = 2x^2 + 3y$ at P(1, 1).

We know that $f$ has minimum directional derivative in the direction of $\hat{u} = \dfrac{-\nabla f(1,1)}{|\nabla f(1,1)|}$ and the value $= -|\nabla f(1,1)|$

Thus, Minimum directional derivative $= -5$

38. Find the directional derivative of the function $f(x, y) = 2x^2 + 3y$ at the point P(1, 1) towards the point (3, -2).

$$\hat{u} = \frac{\langle 3,-2\rangle - \langle 1,1\rangle}{|\langle 3,-2\rangle - \langle 1,1\rangle|} = \frac{\langle 2,-3\rangle}{\sqrt{13}}$$

$$D_{\hat{u}} f(1,1) = \frac{-1}{\sqrt{13}}$$ (same as Q36.)

39. Find the directional derivative of the function $f(x, y, z) = x^2 + yz$ at the point P(1, -3, 2) in the direction of the path $r(t) = t^2\mathbf{i} + 3t\mathbf{j} + (1 - t^3)\mathbf{k}$.

Note at P, $t^2 = 1$, $3t = -3$, $1 - t^3 = 2$

$\Rightarrow t = -1$

$$\Rightarrow \hat{u} = \frac{\vec{r}\,'(-1)}{|\vec{r}\,'(-1)|} = \frac{\langle 2t, 3, -3t^2\rangle\big|_{t=-1}}{|\langle -2, 3, -3\rangle|}$$

$$= \frac{\langle -2, 3, -3\rangle}{\sqrt{22}}$$



$$D_{\hat{u}} f(1,-3,2) = \nabla f(1,-3,2) \cdot \hat{u}$$

$$= \langle 2, 2, -3\rangle \cdot \frac{\langle -2,3,-3\rangle}{\sqrt{22}}$$

$$= \boxed{\frac{11}{\sqrt{22}}}$$ Ans

40. Let $f(x,y) = 2x^2 + 3y^2$ represent the height of a mountain above the point $(x, y)$. Let the positive x-direction point East and the positive y-direction point North. Suppose that I am standing above the point $P(3, 2)$ and sit a ball down. Which direction best represents the direction that the ball will start to roll?

(a) Northwest

(b) Northeast

(c) Southwest

(d) Southeast

(e) None of the above

Hint : Directional derivative

41. Find the rate of change of $f$ with respect to $t$ along the given curve.

$$f(x,y) = x - y,\ r(t) = at\mathbf{i} + b\cos at\mathbf{j}.$$

$x(t) = at$

$y(t) = b\cos at$

direction of the curve = $\vec{r}\,'(t)$

$$\hat{u} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{a\hat{i} - ab\sin at\,\hat{j}}{\sqrt{a^2 + a^2b^2\sin^2 at}}$$

$$D_{\hat{u}} f(at, b\cos at)$$

$$= \nabla f(at, b\cos at) \cdot \hat{u}$$

$$= \langle 1, -1 \rangle \cdot \hat{u} = \frac{a + ab\sin at}{\sqrt{a^2 + a^2b^2\sin^2 at}} \quad \text{Ans}.$$

42. Let $h(t) = f(r(t))$ and $r(t) = \tan t\mathbf{i} + 2\sqrt(2)\sin t\mathbf{j}$. Given that $\nabla f(1, 2) = \mathbf{i} - \mathbf{j}$ and $\nabla f(2, 1) = \mathbf{i} + \mathbf{j}$, find $h'(\pi/4)$.

$$h'(\pi/4) = f'(\vec{r}(\tfrac{\pi}{4}))$$

$$= \nabla f(\vec{r}(\tfrac{\pi}{4})) \cdot \vec{r}\,'(\tfrac{\pi}{4})$$

$$= \nabla f(1, 2) \cdot \langle \sec^2\tfrac{\pi}{4}, 2\sqrt{2}\cos\tfrac{\pi}{4} \rangle$$

$$= (\hat{i} - \hat{j}) \cdot \langle 2, 2 \rangle$$

$$= 2 - 2 = \boxed{0} \quad \text{Ans}$$

43. Answer the following conceptual questions.

(a) When is the directional derivative of $f$ a maximum?

when $\hat{u} = \nabla f \quad (\theta = 0)$

(b) When is it minimum?

$\hat{u} = -\nabla f \quad (\theta = \pi)$

(c) When is it zero?

$\hat{u} \perp \nabla f \quad (\theta = \frac{\pi}{2})$

(d) When is it half of its maximum value?

$D_{\hat{u}} f = \frac{1}{2} |\nabla f| \quad \left(\theta = \frac{\pi}{3}\right)$

when $\hat{u}$ and $\nabla f$ are at an angle $\theta = \frac{\pi}{3}$.

44. Let $f(x,y) = 2xy^2 - \frac{2y}{x}$ and the point P: (1, 2). Determine an equation for the tangent plane to the surface $z = f(x,y)$ at the point (1, 2, 4) on the surface. Also, find an equation of the normal line to the given surface at the indicated point.

Tangent plane:

$$z - 4 = f_x(1,2)(x-1) + f_y(1,2)(y-2)$$

$$\Rightarrow \boxed{z - 4 = 12(x-1) + 6(y-2)}$$

Normal line: Take $F(x,y,z) = f(x,y) - z$

$$\frac{x-1}{f_x(1,2)} = \frac{y-2}{f_y(1,2)} = \frac{z-4}{-1}$$

$$\Rightarrow \frac{x-1}{12} = \frac{y-2}{6} = \frac{z-4}{-1}.$$

45. Let $f(x,y) = 2xy^2 - \frac{2y}{x}$ and the point P: (1, 2).

(a) Calculate the directional derivative of $f$ at the point P in the direction of the vector $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

$$\hat{v} = \frac{3\hat{i} - 4\hat{j}}{\sqrt{9+16}} = \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}$$

$$\nabla f = \left(2y^2 + \frac{2y}{x^2}\right)\hat{i} + \left(4xy - \frac{2}{x}\right)\hat{j}$$

$$\nabla f(1,2) = \langle 12, 6 \rangle$$

$$D_{\hat{u}} f(1,2) = \nabla f(1,2) \cdot \hat{v} = 12\left(\frac{3}{5}\right) - \frac{4}{5}(6) = \frac{12}{5}$$

(b) Determine a unit vector in the direction of the maximum directional derivative of $f$ at P.

$$\hat{u} = \frac{\nabla f(1,2)}{|\nabla f(1,2)|} = \frac{\langle 12, 6 \rangle}{|\langle 12, 6 \rangle|} = \frac{\langle 12, 6 \rangle}{\sqrt{144+36}}$$

$$= \frac{\langle 12, 6 \rangle}{\sqrt{180}} \quad \underline{\text{Ans}}$$

(c) Determine an equation for the tangent plane to the surface $z = f(x,y)$ at the point (1, 2, 4) on the surface.

$$\boxed{z - 4 = 12(x-1) + 6(y-2)} \quad \underline{\text{Ans}}$$

46. Let $f(x, y)$ be the function given below so that $f(3, 0) = 9$. Give the differential approximation to $f(2.9, 0.01)$ where $f(x, y) = x^2e^{xy}$.

$$f_x = 2xe^{xy} + x^2y\,e^{xy} \Rightarrow f_x(3,0) = 6$$

$$f_y = x^3 e^{xy} \Rightarrow f_y(3,0) = 3^3 = 27$$

$$f(x+\Delta x, y+\Delta y) - f(x, y) \approx f_x(x,y)\,\Delta x + f_y(x,y)\,\Delta y$$

$$x = 3 \quad y = 0 \qquad x + \Delta x = 2.9, \quad y + \Delta y = 0.01$$

$$\Rightarrow f(2.9, 0.01) - f(3,0) \approx 6(-0.1) + 27(0.01)$$

$$\approx -0.6 + 0.027$$

$$\Rightarrow f(2.9, 0.01) \approx 9 - 0.6 + 0.027 = \boxed{8.427}$$

47. Find the linear approximation of the function $f(x, y, z) = x^3\sqrt{y^2 + z^2}$ at the point (2, 3, 4) and use it to estimate the number $(1.98)^3\sqrt{(3.01)^2 + (3.97)^3}$.

(Not part of the exam but similar question can be asked for fn of two variables)

Linear approximation is given by an eqn of the tangent plane.

$$f(x,y,z) \approx f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$$

$$= 40 + 3(2)^2\sqrt{3^2+4^2}(x-2) + \frac{2^3(2(3))}{2\sqrt{3^2+4^2}}(y-3) + \frac{2^3(2(4))}{2\sqrt{3^2+4^2}}(z-4)$$

$$= 40 + 12(5)(x-2) + \frac{24}{5}(y-3) + \frac{32}{5}(z-4)$$



$$f(1.98, 3.01, 3.97) \approx 40 + 12(5)(-0.02) + \frac{24}{5}(0.01) + \frac{32}{5}(-0.03)$$

Ans

48. Find the linearization of the function $f(x, y) = \sqrt{x + e^{4y}}$ at the point (3, 0).

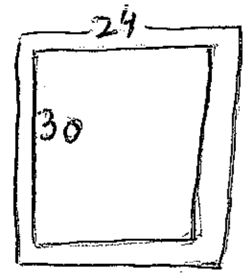
Let the Linearization of f be denoted by L

$$L(x,y) = f(3,0) + f_x(3,0)(x-3) + f_y(3,0)(y-0)$$

$$= 2 + \frac{1}{\sqrt{3+1}}(x-3) + \frac{4e^{4(0)}}{\sqrt{3+1}}(y-0)$$

$$= 2 + \frac{x-3}{2} + 2y = \frac{x}{2} + 2y + \frac{1}{2} \quad \underline{\underline{\text{Ans}}}$$

49. The length and width of a rectangle are measured as 30 cm and 24 cm, respectively, with an error in measurement of at most 0.1 cm in each. Use differentials to estimate the maximum error in the calculated area of the rectangle.




Area of the rectangle $= \underset{\text{length}}{x}\ \underset{\text{width}}{y}$

Let $f(x,y) = xy$

$$\Delta f = f(x+\Delta x, y+\Delta y) - f(x,y) \approx f_x(x,y)\Delta x + f_y(x,y)\Delta y$$

__Note__: $x = 30,\ y = 24\quad \Delta x = 0.1\quad \Delta y = 0.1$

$$\Rightarrow \quad \Delta f = \text{error} \approx 24(0.1) + 30(0.1) = \boxed{5.4}$$

50. Find the second order partial derivatives of $f(x, y) = 4x^3 - xy^2$.

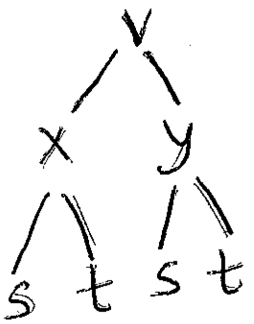
$$f_{xx} = \frac{\partial}{\partial x}(12x^2 - y^2) = 24x$$

$$f_{xy} = \frac{\partial}{\partial y}(12x^2 - y^2) = -2y$$

$$f_{yx} = \frac{\partial}{\partial x}(-2xy) = -2y$$

$$f_{yy} = \frac{\partial}{\partial y}(-2xy) = -2x$$

51. If $v = x^2 \sin y + y e^{xy}$, where $x = s + 2t$ and $y = st$, use the Chain Rule to find $\partial v/\partial s$ and $\partial v/\partial t$ when $s = 0$ and $t = 1$.




$$\frac{\partial v}{\partial s} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial v}{\partial y} \frac{\partial y}{\partial s}$$

$$= (2x \sin y + y^2 e^{xy}) + (x^2 \cos y + e^{xy} + xy e^{xy})\, t$$

When $s=0, t=1$ then $x=2, y=0$

$$\left.\frac{\partial v}{\partial s}\right|_{\substack{s=0,\, t=1 \\ x=2,\, y=0}} = (4 + 1 + 0)1 = 5 \quad \underline{\text{Ans}}$$

Similarly, $$\left.\frac{\partial v}{\partial t}\right|_{s=0,\, t=1} = \left[\frac{\partial v}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial t}\right]_{\substack{s=0,\, t=1 \\ x=2,\, y=0}} = 0(2) + (5)0 = 0 \quad \underline{\text{Ans}}$$

52. Find $\partial z/\partial x$ and $\partial z/\partial y$ given that $x^5 + y^2 + z^3 + xy + zx + yz = \cos(x + y + z)$.

$$f(x,y,z) = x^5 + y^2 + z^3 + xy + xz + yz - \cos(x+y+z) = 0$$

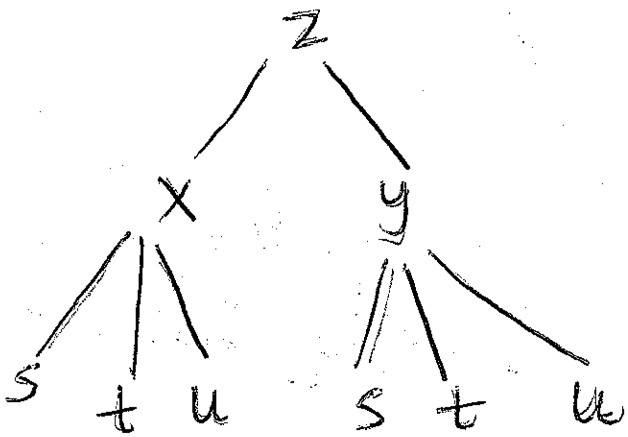
Given that $f(x,y,z) = 0$

$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z} = \frac{-5x^4 + y + z + \sin(x+y+z)}{3z^2 + x + y + \sin(x+y+z)}$$

Similarly $$\frac{\partial z}{\partial y} = -\frac{f_y}{f_z} = \frac{2y + x + z + \sin(x+y+z)}{3z^2 + x + y + \sin(x+y+z)}$$

53. Use the chain rule to find partial derivatives $\frac{\partial z}{\partial u}$, $\frac{\partial z}{\partial s}$, $\frac{\partial z}{\partial t}$ when $u = 1,\ t = 1, s = 0$ given that

$$z = x^2 + xy^3,\ x = s^2t^3 + t^4u + 2u^2,\ y = t^3s + s^2t + u^3.$$




$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$= (2x + y^3)(t^4 + 4u) + (3xy^2)(3u^2)$$

When $u = 1,\ t = 1,\ s = 0$ then

$$x = 3,\ y = 1$$

$$\left.\frac{\partial z}{\partial u}\right|_{u=1, t=1, s=0} = 7(5) + 9(3) = 35 + 27 = \boxed{62}\ \underline{\text{Ans}}$$

Similarly find $\frac{\partial z}{\partial s}$, $\frac{\partial z}{\partial t}$!

54. Determine whether or not the vector function is the gradient, $\nabla f(x,y)$, of a function everywhere defined. If so, find all the functions with that gradient.

$$(xe^{xy} + x^2)\mathrm{i} + (ye^{xy} - 2y)\mathrm{j}$$

Assume: $\nabla f = (x e^{xy} + x^2)\hat{\imath} + (y e^{xy} - 2y)\hat{\jmath}$ ($\to$ FALSE).

then $\frac{\partial f}{\partial x} = x e^{xy} + x^2$ and $\frac{\partial f}{\partial y} = y e^{xy} - 2y$

since differentiating these fn's again would give us continuous fns which means $f$ has continuous second order partial derivative. Therefore by Clairaut's theorem $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ ! But You can check that fail to equal in this case. Our Assumption is incorrect!

55. Determine whether or not the vector function is the gradient, $\nabla f(x,y)$, of a function everywhere defined. If so, find all the functions with that gradient.

$$(y^2e^x - y)\mathrm{i} + (2ye^x - x)\mathrm{j}$$

(Not part of the exam)

Same idea $\frac{\partial f}{\partial x} = y^2 e^x - y$ and $\frac{\partial f}{\partial y} = 2y e^x - x$

But in this case $\frac{\partial^2 f}{\partial x \partial y} = 2y e^x - 1 = \frac{\partial^2 f}{\partial y \partial x} = 2y e^x - 1$

$\Rightarrow$ It is possible that the given fn is the gradient function.

How do we find $f$? It is not going to fit in the space here. Check out the sheet at the end!

56. Find the point(s) on the surface $z = xy$ at which the tangent plane is horizontal.

$f(x,y) = xy$

$f_x(x,y) = y$ and $f_y(x,y) = x$

The points at which the tangent plane is horizontal can be found by solving $\nabla f = 0$

$\Rightarrow f_x = 0$ and $f_y = 0 \Rightarrow y = 0$ and $x = 0$

$\Rightarrow (0,0)$ is the point at which the tangent plane is horizontal.

57. Find the stationary points and the local extreme values.

(a) $f(x,y) = x^2 - 2xy + 2y^2 - 3x + 5y$

I $\nabla f = 0$

$\Rightarrow 2x - 2y - 3 = 0$ and $-2x + 4y + 5 = 0$

Add the two equation $\Rightarrow 2y + 2 = 0 \Rightarrow y = -1 \Rightarrow x = 1/2$

plug in $2x-2y-3=0$

$\Rightarrow (\frac{1}{2}, -1)$ is the critical point.

Classify: $f_{xx}(x,y) = 2$, $f_{xy}(x,y) = -2$, $f_{yy}(x,y) = 4$

$A = f_{xx}(\frac{1}{2}, -1) = 2$, $B = f_{xy}(\frac{1}{2}, -1) = -2$, $C = 4$

and $D = AC - B^2 = 8 - (-2)^2 = 4$.

Note $D > 0$ and $A > 0 \Rightarrow (\frac{1}{2}, -1)$ is a point of minima and the minimum value $= f(\frac{1}{2}, -1)$

$= -13/4$.

(b) $f(x,y) = x^4 - 2x^2 + y^2 - 2$

I $\nabla f = 0 \Rightarrow 4x^3 - 4x = 0$ and $2y = 0$

$\Rightarrow 4x(x^2-1) = 0$ and $y = 0$

$\Rightarrow x = 0, \pm 1$ and $y = 0$.

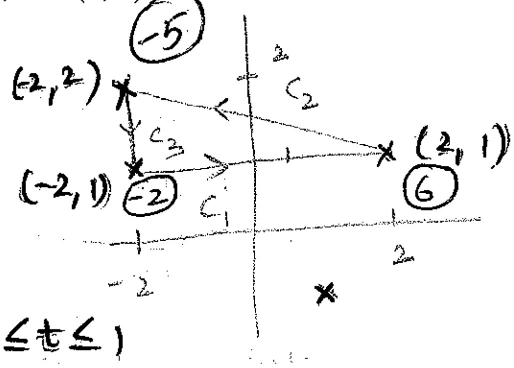
$\Rightarrow (0,0)$, $(0,1)$ $(0,-1)$ are critical points.

Classify: $f_{xx} = 12x^2$ $f_{xy} = 0$ $f_{yy} = 2$

| Points | A | B | C | D | Result |
|---|---|---|---|---|---|
| (0,0) | 0 | 0 | 2 | 0 | Inconclusive |
| (0,1) | 12 | 0 | 2 | 24 | Minima value $= f(1,0) = -3$ |
| (0,-1) | 12 | 0 | 2 | 24 | Minima value $= f(-1,0) = -3$. |

58. Find the absolute extreme values taken on by $f(x, y) = 3 + x - y + xy$ on the set $D$ where

(a) D is the closed triangular region with (-2, 2), (-2, 1), and (2, 1).




I $\nabla f = \langle 1 + y, -1 + x \rangle = \vec{0}$

$\Rightarrow y = -1$ and $x = 1$

$(1, -1)$ is the critical point but it does not belong to the domain. So, Ignore it!

II <u>On $C_1$</u> $x(t) = -2 + t(4)$ $\quad 0 \le t \le 1$

$y(t) = 1 + t(0)$

$f_1(t) = f(x(t), y(t)) = 3 + (-2 + 4t) - 1 + (-2 + 4t)$

No critical points! $f_1(0) = -2$, $f_1(1) = 6$

<u>On $C_2$</u> $x(t) = 2 - t(4)$ $\quad 0 \le t \le 1$

$y(t) = 1 + t$

$f_2(t) = f(x(t), y(t)) = 3 + (2 - 4t) - (1 + t) + (2 - 4t)(1 + t)$

crossing out! Need more space! ← ~~(b) $D = \{(x, y) | x^2 - y^2 \le 4\}$~~

check out the last page...

$f_2'(t) = -4 - 1 + (2 - 4t)(+1) - 4(1 + t) = 0$

$\Rightarrow -5 + 2 - 4t - 4 - 4t = 0$

$\Rightarrow -7 - 8t = 0 \Rightarrow \boxed{t = -\frac{7}{8}}$ ~~this~~ also does not belong to the interval.

Ignore it! $f_2(0) = 6$, $f_2(1) = -5$.

<u>On $C_3$</u> $x(t) = -2 + t(0)$ $\quad 0 \le t \le 1$

$y(t) = 2 - t$

$f_3(t) = f(-2, 2 - t) = 3 - 2 - (2 - t) - 2(2 - t)$

$= 1 - 3(2 - t)$

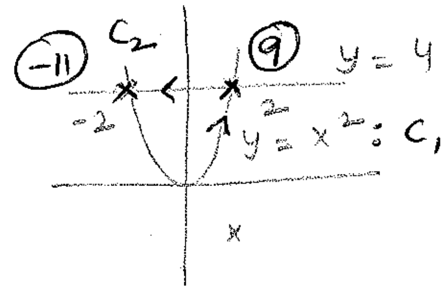
$f_3'(t) \ne 0$ No critical points

$f_3(0) = -5$

$f_3(1) = -2$

Abs max = 6 at (2, 1) and Abs min = -5 at (-2, 2).

(c) D is the region enclosed by $y = x^2$ and $y = 4$.




I $\nabla f = 0$ at $(1, -1)$

Ignore again!

II On $C_1$ : $y = x^2$

$$g(x) = f(x, x^2) = 3 + x - x^2 + x^3, \quad -2 \le x \le 2$$

$$g'(x) = 1 - 2x + 3x^2 = 0$$

$$x = \frac{-(-2) \pm \sqrt{4 - 4(3)(1)}}{2(3)}$$ is not real number

$\Rightarrow$ No critical points!

$$g(-2) = 3 - 2 - 4 - 8 = -11$$

$$g(2) = 3 + 2 - 4 + 8 = 9$$

On $C_2$ : $y = 4$

$$h(x) = f(x, 4) = 3 + x - 4 + 4x, \quad -2 \le x \le 2$$

$$= -1 + 5x$$

Note $h'(x) \neq 0$ No critical points!

$$h(-2) = -1 + 5(-2) = -11$$

$$h(2) = -1 + 5(2) = 9.$$

$\Rightarrow$ Abs. max $= 9$ at $(2, 4)$

Abs. min $= -11$ at $(-2, 4)$. Ans

59. Find the maximum of $f(x,y) = x + y$ on the set where $x^4 + y^4 = 1$ and give the point where this occurs.

Requires Lagrange Method!

(Not a part of Exam II)

Here $g(x,y) = x^4 + y^4$ and $c = 1$

I <u>Set up</u>: $\nabla f = \lambda \nabla g$ and $g(x,y) = 1$

$\langle 1, 1 \rangle = \lambda \langle 4x^3, 4y^3 \rangle$ and $x^4 + y^4 = 1$

$\Rightarrow 4\lambda x^3 = 1, \quad 4\lambda y^3 = 1,$ and $x^4 + y^4 = 1$

II Solve for $x$ and $y$!

Note $\lambda \neq 0$ since $4\lambda x^3 = 1$

$\Rightarrow x^3 = \frac{1}{4\lambda}$ and $y^3 = \frac{1}{4\lambda}$

$\Rightarrow x^3 = y^3 \Rightarrow x = y$

Plug this in the last equation: $x^4 + x^4 = 1$

$\Rightarrow x^4 = \frac{1}{2}$

$\Rightarrow x = \pm \frac{1}{2^{1/4}}$

$\Rightarrow \left(\frac{1}{2^{1/4}}, \frac{1}{2^{1/4}}\right)$ and $\left(\frac{-1}{2^{1/4}}, \frac{-1}{2^{1/4}}\right)$

III $f\left(\frac{1}{2^{1/4}}, \frac{1}{2^{1/4}}\right) = \frac{2}{2^{1/4}}$ (Maximum)

$f\left(\frac{-1}{2^{1/4}}, \frac{-1}{2^{1/4}}\right) = \frac{-2}{2^{1/4}}$ (Minimum)

60. Find the points on the sphere $x^2 + y^2 + z^2 = 1$ that are closest and farthest from the point $(2, 1, 2)$.

Requires Lagrange Method!

(Not a part of Exam II)

I Set up: $g(x,y,z) = x^2+y^2+z^2$ and $c = 1$

$f(x,y,z) = (x-2)^2+(y-1)^2+(z-2)^2$

$\nabla f = \lambda \nabla g$ and $x^2+y^2+z^2 = 1$

$$\left.\begin{aligned} 2(x-2) &= 2\lambda x \\ 2(y-1) &= 2\lambda y \\ 2(z-2) &= 2\lambda z \\ \text{and } x^2+y^2+z^2 &= 1 \end{aligned}\right\} \text{ solve them simultaneously}$$

II solve for $(x, y, z)$

"Fill in yourself"

III Compare the values of $f$ at $(x,y,z)$ found above

61. A package in the shape of a rectangular box can be mailed by the US Postal Service if the sum of its length and girth(the perimeter of a cross section perpendicular to the length) is at most 108 in. Find the dimensions of the package with largest volume that can be mailed.

Not part of the exam II!

Lagrange Method!

Try on your own as a practice problem.

62. Determine the absolute maximum and absolute minimum of

$$f(x,y) = x^2 + 4y^2 - 2x - 4y + 4$$

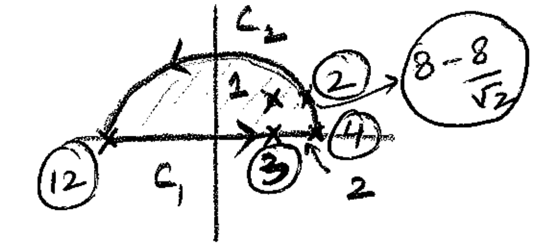
on the region D in the upper half-plane bounded by the x-axis and the ellipse

$$\frac{x^2}{4} + y^2 = 1.$$

I $\nabla f = \langle 2x-2, +8y-4 \rangle = \vec{0}$

$\Rightarrow x = 1$ and $y = 1/2$

$f(1, \frac{1}{2}) = 2$




II On $C_1$ : $y = 0$

$g(x) = f(x,0) = x^2 - 2x + 4, \quad -2 \le x \le 2$

$g'(x) = 2x - 2 = 0 \Rightarrow x = 1$

$g(1) = 3 \quad g(-2) = 12, \; g(2) = 4$

On $C_2$ : $x(t) = 2\cos t, \; y(t) = \sin t, \quad 0 \le t \le \pi$ (half of it)

$h(t) = f(x(t), y(t)) = 4\cos^2 t + 4\sin^2 t - 4\cos t - 4\sin t + 4$

$= 8 - 4\cos t - 4\sin t$

63. Find the maximum value of $F(x,y,z) = 2x + 3y + 5z$ on the sphere

$$x^2 + y^2 + z^2 = 38.$$

Requires Lagrange Method!

(Not a part of the Exam II)

Do it later by yourself after learning Lagrange Method.

continued

$h'(t) = 0$

$\Rightarrow 4\sin t - 4\cos t = 0$

$\Rightarrow \sin t = \cos t$

$\Rightarrow t = \frac{\pi}{4}, \frac{3\pi}{4}$

But $\frac{3\pi}{4}$ does not even belong to $[0, \pi]$

Therefore, we only have one critical point at $t = \pi/4$

$h(\pi/4) = f(\frac{2}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = 8 - \frac{4}{\sqrt{2}} - \frac{4}{\sqrt{2}} = 8 - \frac{8}{\sqrt{2}}$

Note: $h(0) = 4$ and $h(\pi) = 12$.

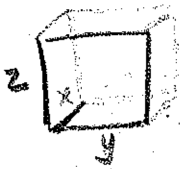


III Compare: Abs max $= 12$ at $(-2, 0)$

Abs min $= 2$ at $(1, \frac{1}{2})$

64. A chemical company plans to construct an open (i.e., no top) rectangular metal tank to hold 256 cubic feet of liquid. It wants to determine the dimensions of the tank that will use the least amount of metal. (Not part of Exam II)

(a) Set up the problem in the form to use the method of Lagrange multipliers, i.e., minimize F subject to the constraint G = c.




Minimize $F(x,y,z) = xy + 2yz + 2zx$ ← surface area with no top.

subject to $xyz = 256$ ← volume

$G(x,y,z) = xyz$

$c = 256.$

(b) Determine the system of equations that has to be solved in order to solve the problem.

$$\nabla F = \lambda \nabla G$$

① $y + 2z = \lambda yz$

② $x + 2z = \lambda xz$

③ $2y + 2x = \lambda xy$

④ $xyz = 256$

55. Continued . . .

$$\frac{\partial f}{\partial x} = y^2 e^x - y \text{ and } \frac{\partial f}{\partial y} = 2ye^x - x$$

$$f(x,y) = \int f_x \, dx \quad (\text{treat } y \text{ as constant})$$

$$= y^2 e^x - xy + \phi(y)$$

$$f_y = \cancel{2ye^x} - \cancel{x} + \phi'(y)$$

$$= \cancel{2ye^x} - \cancel{x} \quad (\text{given})$$

$$\Rightarrow \quad \phi'(y) = 0 \Rightarrow \phi(y) = C \ (\text{constant})$$

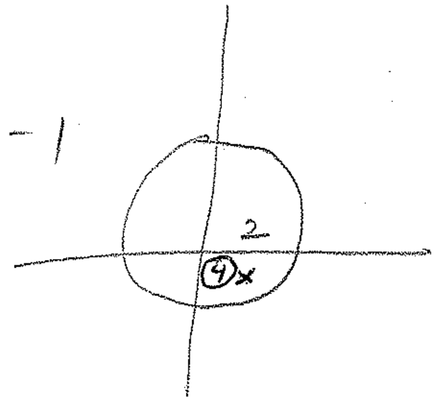
$$\Rightarrow \quad \boxed{f(x,y) = y^2 e^x - xy + C} \quad \underline{\underline{\text{Ans}}}$$

58 (b) D is the closed circular region centered at $(0,0)$ of radius $\leq 2$

$f(x,y) = 3 + x - y + xy$

$\nabla f = 0$ when $x = 1, y = -1$

$(1,-1)$ belongs to the domain.




$f(1,-1) = 3 + 1 + 1 - 1 = 4$

I On the bdry: $x(t) = 2\cos t \quad 0 \leq t \leq 2\pi$
$y(t) = 2\sin t$

$f(t) = f(2\cos t, 2\sin t)$
$= 3 + 2\cos t - 2\sin t + 4\cos t \sin t$
$= 3 + 2(\cos t) - 2\sin t + 2\sin 2t$

$f'(t) = 0$

$\Rightarrow -2\sin t - 2\cos t + 4\cos 2t = 0$

$\Rightarrow 2\cancel{4}\cos 2t = \cancel{2}(\sin t + \cos t)$

Double angle formula $\Rightarrow 2(\cos^2 t - \sin^2 t) = \sin t + \cos t$

$\Rightarrow 2(\cos t - \sin t)(\cos t + \sin t) = \sin t + \cos t$

Factor $\Rightarrow \left[2(\cos t - \sin t) - 1\right](\cos t + \sin t) = 0$

$\Rightarrow \cos t + \sin t = 0$ or $\cos t - \sin t = \frac{1}{2}$

$\Rightarrow \cos t = -\sin t$ or $\cos t = \frac{1}{2} + \sin t$

$\Rightarrow t = \frac{3\pi}{4}, \frac{7\pi}{4}$ or $t =$ very complicated values

$\underline{\text{III}}$ Compare to get Abs max/min

$$f(1,-1) = 4$$

$$f\left(2\cos\frac{\pi}{4}, 2\sin\frac{\pi}{4}\right) = 5$$

$$f\left(2\cos\frac{7\pi}{4}, 2\sin\frac{7\pi}{4}\right) = 5+2\sqrt{2}$$

$$f(\text{complicated values}) = ?$$

$$f(2,0) = 5$$