Exam II Review Problem Set
Math 21-123

1. Find the radius of convergence and the interval of convergence for the following functions:

   (a) $\sum(-1)^k \frac{x^k}{k}$ $\quad a_k = (-1)^k \frac{x^k}{k}$

$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{k}{k+1} |x| = |x| < 1 \Rightarrow$ R.O.C $= 1$, I.O.C $= (-1, 1]$

Check at end points: At $x = -1$, $\sum \frac{1}{k}$ diverges $\because p = 1$

At $x = 1$ $\sum \frac{(-1)^k}{k}$ converges by AST.

   (b) $\sum \frac{k}{6^k} x^k$ $\quad a_k = \frac{k}{6^k} x^k$

$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{k+1}{6k} |x| = \frac{|x|}{6} < 1 \Rightarrow$ R.O.C $= 6$, I.O.C $= (-6, 6)$

Check at end points:

At $x = 6$, $\sum k$ diverges by divergence test.

At $x = -6$, $\sum (-1)^k k$ diverges by divergence test.

   (c) $\sum \frac{k^k}{(2^k)!} x^{2k}$

$a_k = \frac{k^k}{(2^k)!} x^{2k}$, $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{(k+1)^{k+1}}{k^k} \frac{(2^k)!}{(2^{k+1})!} |x|^2$

$= 0 < 1$ for all x.

R.O.C $= \infty$ and I.O.C $= (-\infty, \infty)$.

(d) $\sum(1+\frac{1}{k})^k x^k$

$$a_k = \left(1+\frac{1}{k}\right)^k x^k$$

$$\lim_{k\to\infty} |a_k|^{1/k} = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)|x| = |x| < 1$$

R.O.C = 1 and I.O.C = $(-1, 1)$

<u>Check at end points</u>

At $x=1$, $\sum\left(1+\frac{1}{k}\right)^k$ diverges by divergence test

At $x=-1$, $\sum\left(1+\frac{1}{k}\right)^k(-1)^k$ diverges by DT

(e) $\sum \frac{2^{1/k}\pi^k}{k(k+1)(k+2)}(x-2)^k$

$$a_k = \frac{2^{1/k}\pi^k (x-2)^k}{k(k+1)(k+2)}$$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{2^{1/k+1}\,\pi}{\cancel{(k+1)}\cancel{(k+2)}(k+3)} \frac{k\cancel{(k+1)}\cancel{(k+2)}}{2^{1/k}}|x-2|$$

$$= \pi|x-2| < 1$$

R.O.C = $\frac{1}{\pi}$ and I.O.C = $\left[2-\frac{1}{\pi}, 2+\frac{1}{\pi}\right]$

<u>Check at endpoints</u>

At $x = 2-\frac{1}{\pi}$: $\sum 2^{1/k}\frac{(-1)^k}{k(k+1)(k+2)}$ converges abs by LCT with $b_k = \frac{1}{k^3}$ - convergent by p-series.

At $x = 2+\frac{1}{\pi}$: $\sum \frac{2^{1/k}}{k(k+1)(k+2)}$ converges by the same reasoning

(f) $\sum \frac{k^2}{2\cdot4\cdot6\cdots(2k)}x^k$

$$a_k = \frac{k^2}{2.4\ldots(2k)}x^k = \frac{k^2}{2^k\,k!}x^k$$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{(k+1)^2}{2k^2(k+1)}|x| = 0 < 1 \text{ for all } x.$$

R.O.C $= \infty$

I.O.C $= (-\infty, \infty)$

(g) $\frac{1}{16}(x+1) - \frac{2}{25}(x+1)^2 + \frac{3}{36}(x+1)^3 - \frac{4}{49}(x+1)^4 + \cdots$

$$a_k = \frac{k\,(-1)^{k+1}}{(k+3)^2}(x+1)^k$$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\frac{k+1}{k}\frac{(k+4)^2}{(k+3)^2}|x+1|$$

$$= |x+1| < 1$$

R.O.C $= 1$ and I.O.C $= (-2, 0]$

<u>Check at endpoints</u>: At $x=-2$, $\sum_k \frac{-k}{(k+3)^2}$ divrg. by C.T with $b_k = \frac{1}{k}$.

At $x=0$: $\sum (-1)^{k+1}\frac{k}{(k+3)^2}$ converges by AST.

2. If the radius of convergence of $\sum a_k x^k$ is 8 then can we say anything about the radius of convergence of $\sum a_k x^{3k}$? Why or why not?

R.O.C of $\sum a_K X^{3K}$ is $\sqrt[3]{8} = 2$.

Use Ratio test to conclude above

3. Suppose that the power series $\sum a_k x^k$ converges at $x = 3$ and diverges at $x = -3$. What can you say about the convergence or divergence of the following series?

(a) $\sum a_n 4^n$

Diverges

— div —](— conv —](— div —

-3 0 3

(b) $\sum (-1)^n a_n 2^n$

$= \sum a_n (-2)^n$ converges

4. Suppose that the power series $\sum a_k (x+2)^k$ converges at $x = 4$. At what other values of x must $\sum a_k (x+2)^k$ converge? Does the power series converge at $x = -8$? Explain.

I.O.C = $(-2-R, -2+R]$

$-2+R = 4 \Rightarrow R = 6$

$\sum a_n (x+2)^k$ converges for all x : $-8 < x \le 4$. Nothing can be said about the series at $x = -8$

-8 -2 4 6

5. Find the least value of n for which the sequence of partial sums s_n approximates the sum of the series $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}$ within the error less than 0.0005.

$a_K = \frac{1}{(2K+1)!}$

ASET : $a_{K+1} < 0.0005 \Rightarrow (2K+3)! > 2000$

$\Rightarrow K = 2$ works

$\therefore \sum_{K=0}^{\infty} \frac{(-1)^K}{(2K+1)!} \cong 1 - \frac{1}{3!} + \frac{1}{5!}$ Ans

error < 0.0005

6. Test the series for (a) absolute convergence (b) conditional convergence.

(a) $\sum(-1)^k k \sin\frac{1}{k}$

Note that $\dfrac{\sin\frac{1}{k}}{\frac{1}{k}} \longrightarrow 1$ as $k \to \infty$

$\Rightarrow \sum(-1)^k k \sin\frac{1}{k}$ diverges by divergence test.

(b) $\sum(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}})$

$a_k = \dfrac{1}{\sqrt{k}} - \dfrac{1}{\sqrt{k+1}} = \dfrac{1}{\sqrt{k}\sqrt{k+1}(\sqrt{k}+\sqrt{k+1})} > 0$ (Rationalize)

Use LCT with $b_k = \dfrac{1}{k^{3/2}}$

$\Rightarrow$ The given series converges absolutely.

(c) $\sum \sin(\frac{k\pi}{4})$

Note that $\lim\limits_{k\to\infty} \sin(\frac{k\pi}{4})$ d.n.e.

This implies that $\sum \sin\frac{k\pi}{4}$ diverges by D.T.

(d) $\sum(-1)^k k e^{-k}$

$a_k = (-1)^k k e^{-k}$. Consider $\lim\limits_{k\to\infty} |a_k|^{1/k} = \lim\limits_{k\to\infty} k^{1/k}\frac{1}{e}$

$= \frac{1}{e} < 1$

$\Rightarrow \sum\limits_k (-1)^k k e^{-k}$ converges absolutely by Root test.

(e) $\sum(-1)^k(1-\frac{1}{k})^k$

$$a_k = (-1)^k\left(1-\frac{1}{k}\right)^k$$

Note $\lim_{k\to\infty}\left(1-\frac{1}{k}\right)^k = e^{-1} \neq 0$

$\Rightarrow \sum(-1)^k\left(1-\frac{1}{k}\right)^k$ diverges by DT.

(f) $\sum(-1)^k k^{-(1+1/k)}$

$$a_k = \frac{1}{k^{1+\frac{1}{k}}} > 0 \quad \text{choose } b_k = \frac{1}{k}$$

Note $\lim_{k\to\infty}\frac{a_k}{b_k} = \lim_{k\to\infty}\frac{1}{k^{1/k}} = 1 > 0$ and finite

$\Rightarrow \sum\left|(-1)^k k^{-(1+\frac{1}{k})}\right|$ diverges but $\sum(-1)^k k^{-(1+\frac{1}{k})}$ converges by AST since $a_k \to 0$ and $a_k \downarrow$.

(g) $\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}-\frac{1}{7}+\cdots+\frac{1}{3k+2}-\frac{1}{3k+3}-\frac{1}{3k+4}+\cdots$

$$a_k = \frac{1}{3k+2}-\frac{1}{3k+3}-\frac{1}{3k+4} = \frac{-9k^2-12k-2}{(3k+2)(3k+3)(3k+4)}$$

Note $-a_k > 0$ for all k. Choose $b_k = \frac{1}{k}$

$$\lim_{k\to\infty}\frac{-a_k}{b_k} = \frac{9}{3^3} = \frac{1}{3} > 0 \text{ and finite.}$$

$\Rightarrow \sum -a_k$ diverges by LCT with divergent p-series $b_k = \frac{1}{k}$.

$\Rightarrow \sum a_k$ also diverges.

7. Find a power series representation centered at 0 for the following functions. Also, find R.O.C and I.O.C for the obtained power series.

(a) $x^2 \arctan x$

Note that $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ (Memory)

$x^2 \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{2n+1}$ R.O.C = 1

and I.O.C = $[-1, 1]$

after checking at end points

(b) $\frac{2x}{(1-x^2)}$

$\frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n} \Rightarrow \frac{2x}{1-x^2} = \sum_{n=0}^{\infty} 2x^{2n+1}$

R.O.C = 1 and I.O.C = $(-1, 1)$

after checking at end points.

(c) $\arctan x^2$

$$\arctan x^2 = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}$$

R.O.C = 1 and I.O.C = $[-1, 1]$

after checking at end points

(d) $\frac{x^2+x}{x^2+x-2}$

$$= \frac{x^2+x-2+2}{x^2+x-2} = 1 + \frac{2}{x^2+x-2}$$

(Partial Fraction)
$$= 1 + \frac{2}{3(x-1)} - \frac{2}{3(x+2)}$$

$$\therefore \frac{x^2+x}{x^2+x-2} = 1 + \frac{2}{3}\sum_{n=0}^{\infty} -(x^n) - \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$$
$$= \sum_{n=1}^{\infty}\left(\frac{1-2^{n+1}}{3.2^n}\right)x^n$$

R.O.C = 1 and I.O.C = $\underbrace{(-1,1)}$ does not converge at end points by D.T.

(e) $\frac{x^4-1}{(x-2)^2}$

$$= (x^4-1)\frac{d}{dx}\left(\frac{-1}{x-2}\right)$$
$$= (x^4-1)\frac{d}{dx}\left(+\frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{2^n}\right) = +\frac{1}{2}\sum_{n=1}^{\infty} n\frac{x^{n-1}}{2^n}(x^4-1)$$

$$\Rightarrow \frac{x^4-1}{(x-2)^2} = (x^4-1)\sum_{n=1}^{\infty}\frac{n}{2^{n+1}}(x^{n-1})$$
$$= -\sum_{n=0}^{3}\frac{n+1}{2^{n+2}}x^n + \sum_{n=4}^{\infty}\left(\frac{n-3}{2^{n-2}} - \frac{n+1}{2^{n+2}}\right)x^n.$$

R.O.C = 1 and I.O.C = $[-1, 1]$

converges at end points.

8. Find the Maclaurin series for the following functions.

(a) $\sin x$

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin x$	0
1	$\cos x$	1
2	$-\sin x$	0
3	$-\cos x$	-1
4	$\sin x$	0
$\vdots$	$\vdots$	$\vdots$

Maclaurin Series for $\sin x$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

(b) $\cos x$

Do same as above...

$$\text{Maclaurin Series for } \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

(c) $\cos x^2$

I Use M.S for $\cos x$ from part (b)

$$\text{Maclaurin Series for } \cos x^2 = \sum (-1)^k \frac{x^{4k}}{(2k)!}$$

II - Another way: Draw the table and do it directly.

(d) e^x

Do same as (a)...

$$\text{Maclaurin Series for } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

(e) x^2e^x

Either use part (d) or do it directly.

Maclaurin Series for $x^2e^x = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!}$

R.O.C $= \infty$ and I.O.C $= (-\infty, \infty)$

(f) $\frac{1}{(2+x)^{400}}$

Either use Binomial Series expansion or do it directly.

Maclaurin Series for $\frac{1}{(2+x)^{400}} = \sum_{n=0}^{\infty} \frac{(-1)^n (400) \ldots (400+(n-1))}{2^{400+n}} x^n$

R.O.C $= 1$ and I.O.C $= (-1, 1)$ $= \sum_{n=0}^{\infty} \binom{-400}{n} \frac{x^n}{2^{400+n}}$

diverges at end points.

(g) $\sqrt{1+2x}$

Maclaurin Series for $\sqrt{1+2x} = \sum_{n=0}^{\infty} \binom{1/2}{n} (2x)^n$

$$= \sum_{n=0}^{\infty} \frac{\frac{1}{2}\left(\frac{1}{2}-1\right) \cdots \left(\frac{1}{2}-n+1\right)(2x)^n}{n!}$$

$$= \sum_{n=0}^{\infty} (-1)^{n-1} \frac{1.3.5\ldots(2n-3)(2x)^n}{n!\ 2^n}$$

$$= \sum_{n=0}^{\infty} (-1)^{n-1} \frac{1.3.5\ldots(2n-3)}{n!} x^n$$

R.O.C $= \frac{1}{2}$ and I.O.C $= \left[\frac{-1}{2}, \frac{1}{2}\right]$

9. Prove the following statements.

(a) $\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x.

Look through your class notes

Step I : Find Maclaurin Series for $\cos x$.

Step II : show that the Remainder goes to zero as $n \longrightarrow \infty$.

$$R_n(x) = f^{(n+1)}(z) \frac{x^{n+1}}{(n+1)!}$$

for some z between 0 and x.

(b) $\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$ for all x.

Same as part a ...

(c) $e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$ for all x.

Same as part (a)...

(d) $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$ for all $-1 < x < 1$.

Step I. Write power series for $\frac{1}{1+x}$

Step II. Integrate both sides.

DONE!

(e) $\arctan x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$ for all $-1 < x < 1$.

Step I. Write power series representation of ~~arctan~~ $\frac{1}{1+x^2}$ centered at 0.

Step II. Integrate both sides

DONE!

10. Find the sum of the following series.

(a) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

'n' in the numerator indicates that it is the derivative of some power series.

Note: $\frac{1}{(1-x)^2} = \left(\frac{1}{1-x}\right)' = \left(\sum_{n=0}^{\infty} x^n\right)' = \sum_{n=1}^{\infty} n x^{n-1}$

Plug $x = \frac{1}{2}$ $\quad 2^2 = \sum_{n=1}^{\infty} \frac{n}{2^{n-1}} \Rightarrow 2^2 = \sum_{n=1}^{\infty} \frac{n}{2^n} \cdot 2 \Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{n}{2^n} = 2}$

(b) $\sum_{n=2}^{\infty} \frac{n^2-n}{2^n}$ indicates second derivative

$= \sum_{n=2}^{\infty} \frac{n(n-1)}{2^n}$ — indicates second derivative

Note: $\frac{2}{(1-x)^3} = \left(\sum_{n=0}^{\infty} x^n\right)'' = \sum_{n=2}^{\infty} n(n-1) x^{n-2}$

Plug $x = \frac{1}{2}$ $\quad 2(2^3) = \sum_{n=2}^{\infty} \frac{n(n-1)}{2^{n-2}} \Rightarrow \boxed{\sum_{n=2}^{\infty} \frac{n(n-1)}{2^n} = 4}$

(c) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Note: $\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \sum_{n=1}^{\infty} \left(\frac{n^2-n}{2^n} + \frac{n}{2^n}\right)$

Both series converge $\Leftarrow = \sum_{n=1}^{\infty} \left(\frac{n^2-n}{2^n}\right) + \sum_{n=1}^{\infty} \frac{n}{2^n}$

change of index $\Leftarrow = \sum_{n=2}^{\infty} \frac{n^2-n}{2^n} + \sum_{n=1}^{\infty} \frac{n}{2^n}$

$= \underset{\text{part (b)}}{4} + \underset{\text{part (a)}}{2} = \boxed{6}$

(d) $\sum_{k=1}^{\infty} \frac{1}{k!}$

Note: It resembles the power series for e^x with $x = 1$. $\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$ for all x.

In particular take $x = 1$: $\sum_{k=0}^{\infty} \frac{1}{k!} = e \Rightarrow \boxed{\sum_{k=1}^{\infty} \frac{1}{k!} = e-1}$

(e) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$

Note: $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x.

In particular take $x = 1$: $\boxed{\sin 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}}$

(f) $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{(2n+1)!}$

$= \sin \pi = \boxed{0}$ Ans

(g) $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{2^{2n}(2n)!}$

$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{\pi}{2}\right)^{2n} \frac{1}{(2n)!} = \cos \frac{\pi}{2} = \boxed{0}$ Ans

(h) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n2^n}$

indicates integral

Transition into x

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} = -\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{n} = \int\sum_{n=0}^{\infty}(-x)^n\,dx = \int\frac{1}{1+x}\,dx$$

$$= \ln(1+x)$$

Plug in $x=\frac{1}{2}$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n2^n} = \ln\left(1+\frac{1}{2}\right) = \ln\frac{3}{2} \quad \underline{\underline{\text{Ans}}}.$$

(i) $\sum_{n=1}^{\infty}\frac{1}{n2^n}$

$$= -\ln\left|1-\frac{1}{2}\right| = -\ln\frac{1}{2} = \ln 2 \quad \underline{\underline{\text{Ans}}}.$$

Note that $\quad -\ln(1-x) = \int\frac{1}{1-x}\,dx = \int\sum_{n=0}^{\infty}x^n\,dx$

$$= \sum_{n=1}^{\infty}\frac{x^n}{n}$$

(j) $\sum_{n=1}^{\infty}\frac{3^n}{n!5^n}$

$$= \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{3}{5}\right)^n = e^{3/5}-1 \quad \underline{\underline{\text{Ans}}}$$

<u>Note</u>: $\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{3}{5}\right)^n = e^{3/5}$.

11. Find the Taylor series for $f(x)$ centered at the given value of c.

(a) $f(x) = e^x$, $c = 3$

T.S centered at 3 $= \sum e^3 \frac{(x-3)^n}{n!}$

derive this by computing the values of the derivative of e^x at $x = 3$.

(b) $f(x) = \ln x$ $c = 1$

T.S centered at 1 $= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n!} (n-1)!$

$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}$ Ans

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln x$	0
1	$1/x$	1
2	$-1/x^2$	-1
3	$2/x^3$	$2(1)$
4	$-\frac{3(2)}{x^4}$	$-3(2)$
5	$\frac{4(3)(2)}{x^5}$	$4(3)(2)$
⋮	⋮	⋮

Alternating.

(c) $f(x) = \sin x \quad c = \frac{\pi}{2}$

n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$
0	$\sin x$	1
1	$\cos x$	0
2	$-\sin x$	-1
3	$-\cos x$	0
4	$\sin x$	1
5	$\cos x$	0
$\vdots$	$\vdots$	$\vdots$

T. S centered at $\pi/2$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{(x-\pi/2)^{2k}}{(2k)!} \quad \underline{\underline{\text{Ans}}}$$

(d) $f(x) = x^{-2} \quad c = 1$

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	x^{-2}	$1 = 1!$
1	$-2x^{-3}$	$-2 = -2!$
2	$2(3)x^{-4}$	$2(3) = 3!$
3	$-4(3)(2)x^{-5}$	$-4(3)(2) = -4!$
4	$5(4)(3)(2)x^{-6}$	$5(4)(3)(2) = 5!$
5	$-6(5)(4)(3)(2)x^{-7}$	$-6(5)(4)(3)(2) = -6!$
6	$\vdots$	$\vdots$
$\vdots$		

T. S centered at 1

$$= \sum_{n=0}^{\infty} (-1)^n (n+1)! \frac{(x-1)^n}{n!}$$

$$= \sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n$$

$\underline{\underline{\text{Ans}}}$

12. Use series to compute the following integral.

(a) $\int \frac{\sin x}{x}\, dx$

Note: $\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$

$$\frac{\sin x}{x} = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k+1)!}$$

$$\int \frac{\sin x}{x}\, dx = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)(2k+1)!} + C$$

(b) $\int e^{-x^2}\, dx$

Note: $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$

$$\int e^{-x^2} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{2n+1} + C.$$

(c) $\int \frac{e^x - 1}{x}\, dx$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow \frac{e^x - 1}{x} = \frac{1}{x} \sum_{n=1}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$$

$$\int \frac{e^x - 1}{x}\, dx = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)\,(n+1)!} + C \quad \underline{\underline{\text{Ans}}}$$

(d) $\int \frac{1-\cos x}{x}\, dx$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad \text{for all } x$$

$$\frac{1-\cos x}{x} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n)!}$$

$$\int \frac{1-\cos x}{x}\, dx = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n}}{2n\,(2n)!} + C \quad \underline{\underline{\text{Ans}}}$$

13. Use series to approximate the value of the following definite integrals within the specified error.

(a) $\int_0^1 e^{-x^2}\,dx$ within 0.0001.

Note that $$\int_0^1 e^{-x^2}dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\,n!}\Bigg]_0^1$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\,n!}$$

<u>I way</u> <u>Use ASET</u>:

This is a convergent alternating series with decreasing terms (✓/✓/✓). Therefore by ASET

$$\frac{1}{(2n+3)\,(n+1)!} < 0.0001$$

$$\Rightarrow \quad 10000 < (2n+3)\,(n+3)!$$

$n = 4$ is the smallest integer that satisfies the above inequality.

$$\int_0^1 e^{-x^2}dx \underset{\substack{\uparrow\\ \text{error} = 0.0001}}{\approx} T_4(1) = 1 - \frac{1}{3(1!)} + \frac{1}{5(2!)} - \frac{1}{7(3!)} + \frac{1}{9(4!)}$$

<u>II way</u> <u>Use Remainder formula</u>:

— very difficult to find a pattern to get $(n+1)^{th}$ derivative

— Not advisable!

(b) $\int_0^1 e^{x^2}\,dx$ within 0.0004. — Not ideal for exam

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

$$\int_0^1 e^{x^2}dx = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!\,(2n+1)}\Bigg]_0^1$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!\,(2n+1)}$$

There is only one way to find the value of n for this series since it is a positive term series — Use Taylor's formula.

$$R_n(x) = \underbrace{f^{(n+1)}(z)}_{\uparrow}\,\frac{x^{n+1}}{(n+1)!}$$

very difficult to compute as shown below:

$f(x) = e^{x^2}$, $f'(x) = 2xe^{x^2}$, $f''(x) = (2x)^2e^{x^2} + 2e^{x^2}$

$f'''(x) = (2x)^3e^{x^2} + 8xe^{x^2} + 2(2x)e^{x^2} \ldots$

product makes it difficult to see the pattern. This question illustrates that ASET makes such app. handy given that the series is alternating.

14. Use series to compute the following limits.

(a) $\lim_{x\to 0} \frac{x - \arctan x}{x^3}$

$$x - \arctan x = \cancel{x} - \left(\cancel{x} \mp \frac{x^3}{3} \pm \frac{x^5}{5} \mp \cdots\right)$$

$$\lim_{x\to 0} \frac{x - \arctan x}{x^3} = \lim_{x\to 0} \frac{1}{3} - \frac{x^2}{5} - \cdots$$

$$= \boxed{\frac{1}{3}}$$

(b) $\lim_{x\to 0} \frac{1-\cos x}{x}$

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{x - \left(\cancel{1} - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)}{x}$$

$$= \lim_{x\to 0} \frac{\frac{x^2}{2!} - \frac{x^4}{4!} + \cdots}{x}$$

$$= \lim_{x\to 0} \frac{x}{2!} - \frac{x^3}{4!} + \cdots$$

$$= \boxed{0}$$

15. Use series to solve the following differential equations.

(a) $f'(x) = f(x)$ and $f(0) =$ [redacted]

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$

Note that $f'(x) = f(x)$ implies that

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow a_{n+1}(n+1) = a_n \text{ for all } n \geq 0$$

$$\Rightarrow \boxed{a_{n+1} = \frac{a_n}{n+1}}$$

$$f(0) = 1 \Rightarrow a_0 = 1$$

$$\Rightarrow a_1 = \frac{a_0}{0+1} = 1$$

$$\Rightarrow a_2 = \frac{a_1}{1+1} = \frac{1}{2}$$

$$\vdots$$

$$a_n = \frac{1}{n!}$$

$$\Rightarrow \boxed{f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}} \text{ for all } x. \Rightarrow \boxed{f(x) = e^x}$$

(b) $f''(x) + f(x) = 0$, $f'(0) = 0$, and $f(0) = 1$

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$

Note that $f''(x) + f(x) = 0$

$$\Rightarrow \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \quad a_{n+2} = \frac{-a_n}{(n+1)(n+2)} \quad \text{for all } n \geq 0.$$

$$f'(0) = 0 \Rightarrow a_1 = 0$$

$$f(0) = 1 \Rightarrow a_0 = 1$$

$$\Rightarrow a_2 = -\frac{a_0}{2} = \frac{-1}{2} = \frac{-1}{2!}$$

$$\Rightarrow a_3 = 0 \Rightarrow a_5 = 0 \Rightarrow \cdots \Rightarrow a_{2n+1} = 0$$

for all $n \geq 0$.

and $a_4 = \frac{-a_2}{3(4)} = \frac{1}{4!}$

$$\Rightarrow \cdots \Rightarrow a_{2n} = \frac{(-1)^n}{(2n)!} \quad \text{for all } n \geq 0.$$

$$\Rightarrow \boxed{f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}} \quad \text{for all } x.$$

$$\Rightarrow \boxed{f(x) = \sin x}$$ Ans.

(c) $f''(x) + f(x) = 0,\ f'(0) = 1$, and $f(0) = 0$

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

Note that $f''(x) + f(x) = 0$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} \left((n+1)(n+2) a_{n+2} + a_n\right) x^n = 0$$

$$\Rightarrow \quad a_{n+2} = -\frac{a_n}{(n+1)(n+2)} \quad \text{for all } n \geq 0.$$

$f'(0) = 1 \Rightarrow a_1 = 1$

$f(0) = 0 \Rightarrow a_0 = 0$

$a_2 = \frac{a_0}{2} = 0 \ \ldots \ a_{2n} = 0$ for all n.

$a_3 = -\frac{a_1}{2(3)} = \frac{-1}{3!} \ \ldots \ a_{2n+1} = \frac{(-1)^n}{(2n+1)!}$ for all n.

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sin x} \quad (Q\ 9(b))$$

Ans

16. Let $f(x) = \sqrt{x}$.

(a) Find the Taylor polynomial of degree 2 for f at $c = 4$.

n	$f^{(n)}(x)$	$f^{(n)}(4)$
0	$\sqrt{x}$	2
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{2(2)}$
2	$\frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2}$	$\frac{1}{2(-2)(2^3)}$

$$T_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{16}(x-2)^2$$

Ans

(b) How accurately does the Taylor polynomial obtained in (a) approximates the function f when $3 \le x \le 5$.

$$R_2(x) = f^{(3)}(z)\frac{(x-4)^{n+1}}{(n+1)!}$$ where z lies b/w 4 and x.

$$|R_2(x)| = |f^{(3)}(z)| \frac{|x-4|^3}{3!}$$

$$= \left|\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)z^{-5/2}\right| \frac{|x-4|^3}{3!}$$

$3 \le x \le 5$

$\Rightarrow |x-4| \le 1$

$$\le \frac{3}{8(3!)} \underbrace{\max_z \left|\frac{1}{z^{5/2}}\right|}_{\le 1} \frac{|x-4|^3}{3!}$$

b/c either $4 < z < x \le 5$ or $3 \le x < z < 4$

$$\le \frac{1}{16}$$

error. $= \boxed{0.0625}$

17. Let $f(x) = \frac{e^x - 1}{x}$.

(a) Expand f as a power series.

$$\frac{e^x - 1}{x} = \frac{\sum_{n=0}^{\infty} \frac{x^n}{n!} - 1}{x} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$$

(b) Differentiate the power series obtained in part (a) and show that $\sum_1^\infty \frac{n}{(n+1)!} = 1$.

Note: $$\left(\frac{e^x - 1}{x}\right)' = \left(\sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}\right)'$$

$$\Rightarrow \frac{xe^x - e^x + 1}{x^2} = \sum_{\substack{n=2 \\ n-1=1}}^{\infty} (n-1)\frac{x^{n-2}}{n!}$$

$$\Rightarrow \frac{xe^x - e^x + 1}{x^2} = \sum_{n=1}^{\infty} \frac{n}{(n+1)!} x^{n-1}$$ change of index

Plug in $x = 1$ $$\Rightarrow \frac{e - e + 1}{1} = \sum_{n=1}^{\infty} \frac{n}{(n+1)!}$$

$$\Rightarrow \boxed{1 = \sum_{n=1}^{\infty} \frac{n}{(n+1)!}}$$ Ans

18. Let $f(x) = xe^x$.

(a) Expand f as a power series.

$$\boxed{xe^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}} \text{ for all } x.$$

(b) Integrate the power series obtained in part (a) and show that $\sum_1^\infty \frac{n+1}{(n+2)!} = \frac{1}{2}$.

$$\int xe^x = \int \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}\, dx + C$$

I.P $u = x$

$dv = e^x dx$

$$\Rightarrow \quad xe^x - e^x = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!\,(n+2)} + C$$

Plug in $x = 0$ to get the value of C.

$$-1 = 0 + C \Rightarrow \boxed{C = -1}$$

Plug in $x = 1$ to get the value of the sum.

$$e - e = \sum_{n=0}^{\infty} \frac{1}{n!(n+2)} - 1$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{1}{n!\,(n+2)} = 1 \Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{1}{n!\,(n+2)} = 1 - \frac{1}{2} = \frac{1}{2}}$$

19. Assume that f is a function such that $|f^{(n)}(x)| < 1$ for all n and x.

(a) Estimate the error if $T_5(1/2)$ is used to approximate $f(1/2)$.

$$\left|R_5\left(\tfrac{1}{2}\right)\right| \leq \max_z |f^{(6)}(z)| \frac{|x|^6}{6!} \quad \sqrt{x} = \tfrac{1}{2}$$

$$\leq \boxed{\left(\frac{1}{2}\right)^6 \frac{1}{6!}} \rightarrow \text{error.}$$

(b) Find the least positive integer n for which $T_n(-4)$ approximates $f(-4)$ within 0.001.

$$|R_n(-4)| \leq \max_z |f^{(n+1)}(z)| \frac{|-4|^{n+1}}{(n+1)!}$$

$$\leq \frac{4^{n+1}}{(n+1)!} < 0.001$$

$$\Rightarrow \frac{(n+1)!}{4^n} > 4000 \Rightarrow \boxed{n = 14}$$ is the smallest positive integer that works.

(c) Find the values of x such that the error in the approximation of f by T_2 is less than 0.001.

$$|R_2(x)| \leq \max_z |f^{(2+1)}(z)| \frac{|x|^3}{3!}$$

$$\leq \frac{|x|^3}{3!} < 0.001$$

$$\Rightarrow |x|^3 < \frac{3!}{1000} \Rightarrow \boxed{|x| < \frac{\sqrt[3]{6}}{10}} \text{ Ans}$$

20. Let $f(x) = e^x$.

(a) Determine the maximum possible error we incur by using $T_6(x)$ to approximate $f(x) = e^x$ for x in $[0, 1]$.

$$e^x = 1 + x + \frac{x^2}{2!} + \dots \quad \text{for all } x.$$

Fix x in the interval $[0,1]$.

$$T_6(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^6}{6!}$$

$$|R_6(x)| \le \max_{0 \le z < x} |f^{(7)}(z)| \frac{|x|^7}{7!} \underset{|x|<1}{\le} \frac{e^x}{7!} \underset{e^x < e^1}{\le} \frac{e}{7!} \underset{e<3}{\le} \frac{3}{7!}$$

$$\boxed{\text{Max possible error} < \frac{3}{7!} = 0.0006}$$

(b) Give an estimate $e^{0.2}$ correct to three decimal places, that is, remainder is less than 0.0005.

$$T_n(0.2) = 1 + 0.2 + \frac{(0.2)^2}{2!} + \dots$$

$$R_n(0.2) \le e^{0.2} \frac{|0.2|^{n+1}}{(n+1)!} < 3 \frac{1}{5^{n+1}(n+1)!}$$

$$0 < z < 0.2$$

$$e^0 < e^z < e^{0.2}$$

$$< 0.0005 = \frac{5}{10000}$$

$\Rightarrow \quad 6000 < 5^{n+1}(n+1)!$

$\Rightarrow \quad n = 3$ – smallest integer that works.

Ans $$\boxed{T_3(0.2) = 1 + 0.2 + \frac{(0.2)^2}{2!} + \frac{(0.2)^3}{3!} \sim 1.22133}$$

21. Estimate $\sin 0.5$ within 0.0001 using

(a) Lagrange's Remainder

$$|R_n(0.5)| \le \max_z |f^{(n+1)}(z)| \frac{(0.5)^{n+1}}{(n+1)!}$$

$$\le \frac{(0.5)^{n+1}}{(n+1)!} < 0.0001$$

$\Rightarrow 2^{n+1}(n+1)! > 10000 \Rightarrow n = 6$ is the smallest integer that works.

$$\sin 0.5 \underset{\uparrow}{\cong} T_6(0.5) \underset{\uparrow}{=} T_5(0.5)$$

error < 0.0001 because even terms are zero.

(b) Alternating Series Estimation theorem

$$\sin 0.5 = 0.5 - \frac{(0.5)^3}{3!} + \frac{(0.5)^5}{5!} - \ldots$$

This is a convergent alternating series with decreasing terms. Therefore by ASET,

$$\frac{(0.5)^n}{n!} < 0.0001$$

$$\Rightarrow 10000 < 2^{n+1}(n+1)!$$

$\Rightarrow n = 6$ is the smallest integer that works.

$T_6(0.5)$ approximates $\sin(0.5)$ within 0.0001.

22. Estimate $\ln(1.4)$ and $\sqrt{e}$ to within 0.01.

I $\ln(1.4) = \ln(1+0.4) = 0.4 - \frac{1}{2}(0.4)^2 + \frac{(0.4)^3}{3!} - \ldots$

This is a convergent alternating series with decreasing terms. Therefore by ASET

$$\frac{(0.4)^{n+1}}{n+1} < 0.01$$

$$\Rightarrow \quad 100 < \frac{(n+1)10^{n+1}}{4^{n+1}}$$

$\uparrow$ $n = 3$ works.

$$\therefore \ln(1.4) \approx 0.4 - \frac{(0.4)^2}{2} + \frac{(0.4)^3}{3}$$

II Same as Q 20 (b). – Use Remainder formula.

$$|R_n(0.5)| < \max_{0<z<0.5} e^z \frac{|0.5|^{n+1}}{(n+1)!}$$

$\max e^z < e^{0.5} < 3$

$$< \frac{3}{(2^{n+1})(n+1)!} < 0.01$$

$$\Rightarrow \quad 300 < 2^{n+1}(n+1)!$$

$\Rightarrow$ $\boxed{n = 3}$ is the smallest integer that works.

$$\boxed{\sqrt{e} \approx 1 + 0.5 + \frac{(0.5)^2}{2} + \frac{(0.5)^3}{3!}} = 1.645833$$

Ans.