Exam II Review Problem Set
Math 21-123

---

1. Find the radius of convergence and the interval of convergence for the following functions:

   (a) $\sum(-1)^k \frac{x^k}{k}$
   (b) $\sum \frac{k}{6^k} x^k$
   (c) $\sum \frac{k^k}{(2^k)!} x^{2k}$
   (d) $\sum(1+\frac{1}{k})^k x^k$
   (e) $\sum \frac{2^{1/k}\pi^k}{k(k+1)(k+2)}(x-2)^k$
   (f) $\sum \frac{k^2}{2\cdot 4\cdot 6\cdots(2k)} x^k$
   (g) $\frac{1}{16}(x+1) - \frac{2}{25}(x+1)^2 + \frac{3}{36}(x+1)^3 - \frac{4}{49}(x+1)^4 + \cdots$

2. If the radius of convergence of $\sum a_k x^k$ is 8 then can we say anything about the radius of convergence of $\sum a_k x^{3k}$? Why or why not?

3. Suppose that the power series $\sum a_k x^k$ converges at $x = 3$ and diverges at $x = -3$. What can you say about the convergence or divergence of the following series?

   (a) $\sum a_n 4^n$
   (b) $\sum(-1)^n a_n 2^n$

4. Suppose that the power series $\sum a_k(x+2)^k$ converges at $x = 4$. At what other values of x must $\sum a_k(x+2)^k$ converge? Does the power series converge at $x = -8$? Explain.

5. Find the least value of n for which the sequence of partial sums s_n approximates the sum of the series $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}$ within the error less than 0.0005.

6. Test the series for (a) absolute convergence (b) conditional convergence.

   (a) $\sum(-1)^k k \sin\frac{1}{k}$
   (b) $\sum(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}})$
   (c) $\sum \sin(\frac{k\pi}{4})$
   (d) $\sum(-1)^k k e^{-k}$
   (e) $\sum(-1)^k(1-\frac{1}{k})^k$
   (f) $\sum(-1)^k k^{-(1+1/k)}$
   (g) $\frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \cdots + \frac{1}{3k+2} - \frac{1}{3k+3} - \frac{1}{3k+4} + \cdots$

7. Find a power series representation for the following functions. Also, find R.O.C and I.O.C for the obtained power series.

(a) $x^2 \arctan x$

(b) $\frac{2x}{(1-x^2)}$

(c) $\arctan x^2$

(d) $\frac{x^2+x}{x^2+x-2}$

(e) $\frac{x^4-1}{(x-2)^2}$

8. Find the Maclaurin series for the following functions.

(a) $\sin x$

(b) $\cos x$

(c) $\cos x^2$

(d) e^x

(e) $x^2 e^x$

(f) $\frac{1}{(2+x)^{400}}$

(g) $\sqrt{1+2x}$

9. Prove the following statements.

(a) $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x.

(b) $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ for all x.

(c) $e^x = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$ for all x.

(d) $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ for all $-1 < x < 1$.

(e) $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ for all $-1 < x < 1$.

10. Find the sum of the following series.

(a) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

(b) $\sum_{n=2}^{\infty} \frac{n^2-n}{2^n}$

(c) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

(d) $\sum_{k=1}^{\infty} \frac{1}{k!}$

(e) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$

(f) $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{(2n+1)!}$

(g) $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{2^{2n}(2n)!}$

(h) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n2^n}$

(i) $\sum_{n=1}^{\infty} \frac{1}{n2^n}$

(j) $\sum_{n=1}^{\infty} \frac{3^n}{n!5^n}$

11. Find the Taylor series for $f(x)$ centered at the given value of c.

   (a) $f(x) = e^x, \quad c = 3$
   (b) $f(x) = \ln x, \quad c = 1$
   (c) $f(x) = \sin x, \quad c = \frac{\pi}{2}$
   (d) $f(x) = x^{-2}, \quad c = 1$

12. Use series to compute the following integral.

   (a) $\int \frac{\sin x}{x}\, dx$
   (b) $\int e^{-x^2}\, dx$
   (c) $\int \frac{e^x - 1}{x}\, dx$
   (d) $\int \frac{1-\cos x}{x}\, dx$

13. Use series to approximate the value of the following definite integrals within the specified error.

   (a) $\int_0^1 e^{-x^2}\, dx$ within 0.0001.
   (b) $\int_0^1 e^{x^2}\, dx$ within 0.0004.

14. Use series to compute the following limits.

   (a) $\lim_{x\to 0} \frac{x - \arctan x}{x^3}$
   (b) $\lim_{x\to 0} \frac{1-\cos x}{x}$

15. Use series to solve the following differential equations.

   (a) $f'(x) = f(x)$ and $f(0) = 1$
   (b) $f''(x) + f(x) = 0, \; f'(0) = 0$, and $f(0) = 1$
   (c) $f''(x) + f(x) = 0, \; f'(0) = 1$, and $f(0) = 0$

16. Let $f(x) = \sqrt{x}$.

   (a) Find the Taylor polynomial of degree 2 for f at $c = 4$.
   (b) How accurately does the Taylor polynomial obtained in (a) approximates the function f when $3 \le x \le 5$.

17. Let $f(x) = \frac{e^x - 1}{x}$.

   (a) Expand f as a power series centered at c = 0.
   (b) Differentiate the power series obtained in part (a) and show that $\sum_1^\infty \frac{n}{(n+1)!} = 1$.

18. Let $f(x) = xe^x$.

   (a) Expand f as a power series centered at c = 0.

(b) Integrate the power series obtained in part (a) and show that $\sum_1^\infty \frac{n+1}{(n+2)!} = \frac{1}{2}$.

19. Assume that f is a function such that $|f^{(n)}(x)| < 1$ for all n and x.

    (a) Estimate the error if $T_5(1/2)$ is used to approximate $f(1/2)$.

    (b) Find the least positive integer n for which $T_n(-4)$ approximates $f(-4)$ within 0.001.

    (c) Find the values of x such that the error in the approximation of f by T_2 is less than 0.001.

20. Let $f(x) = e^x$.

    (a) Determine the maximum possible error we incur by using $T_6(x)$ to approximate $f(x) = e^x$ for x in $[0, 1]$.

    (b) Give an estimate $e^{0.2}$ correct to three decimal places, that is, remainder is less than 0.0005.

21. Estimate $\sin 0.5$ within 0.0001 using

    (a) Lagrange's Remainder

    (b) Alternating Series Estimation theorem

22. Estimate $\ln(1.4)$ and $\sqrt{e}$ to within 0.01.