Specifics of Exam III

1. Date: April 22, Friday
2. Duration: 50 minutes
3. Venue: In-class
4. Syllabus: Section 11.8 and Chapter 12
5. Extra Office Hours: Th 2-4 pm
6. Instruction:
   (a) Arrive at least 5 minutes prior to the scheduled time.
   (b) Remember points are awarded for showing work and an understanding of the concepts.
   (c) Practice problems = Hw + Quiz + Few Extra problems + Problem from the Summary sheet of Chapter 12.
   (d) If you see (Hw) written along with the problem in this review sheet then it means the problem is from the homework, (Quiz) means from the quiz, and (DI-extra.pdf) means from the summary of chapter 12 posted on the lecture page.

List and Brief Description of Important Topics

1. **Lagrange's method of Multipliers** is used to solve the following constraint optimization problem,

   $$\text{Maximize/Minimize } f(x,y,z) \text{ subject to the constraint } g(x,y,z) = k.$$

   According to this method, we solve the following set of equation for (x,y,z) to get the candidates for the points that give us the maximum or the minimum values of the function f.

   $$\begin{aligned} f_x(x,y,z) &= \lambda g_x(x,y,z) \\ f_y(x,y,z) &= \lambda g_y(x,y,z) \\ f_z(x,y,z) &= \lambda g_z(x,y,z) \\ g(x,y,z) &= k \end{aligned}$$

   Things to Remember:

   (a) The constraint must be of the form $g(x,y,z) = k$, so you may have to rewrite the constraint.

   (b) There is usually more than one way to solve the equations.

   (c) Be careful while dividing by variables. Do not divide by ZERO.

   (d) A maximum or minimum may not exist.

2. **Double Integrals:** We defined double integral in a similar way to single variable integral using partitions of the domain which is usually very hard to work with for computational purposes so we use iterated integrals. Below are defined iterated integrals over various regions.

   (a) If a region R is a rectangle of the form $\{(x,y)|a \le x \le b,\ c \le y \le d\}$ then

   $$\iint_R f(x,y)\,\mathrm{d}A = \int_c^d \int_a^b f(x,y)\,\mathrm{d}x\mathrm{d}y = \int_c^d \int_a^b f(x,y)\,\mathrm{d}x\mathrm{d}y.$$

   In this case, the order of integration can be interchanged without hesitation.

   (b) If a region R has the form $\{(x,y)|a \le x \le b; g_1(x) \le y \le g_2(x)\}$, then the integral over the region is given by

   $$\iint_R f(x,y)\,\mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y\mathrm{d}x.$$

   (c) Similarly if $R = \{(x,y)|h_1(y) \le x \le h_2(y); c \le y \le d\}$, then the integral is given by

   $$\iint_R f(x,y)\,\mathrm{d}A = \int_c^d \int_{h_1(y))}^{h_2(y)} f(x,y)\,\mathrm{d}x\mathrm{d}y.$$

Application of Double integrals: In the same way that integration can be used to find areas in one-variable calculus we can use double integrals to find volumes bounded by two-dimensional surfaces. If surface $z = f(x, y)$ lies above the surface $z = g(x, y)$ then the volume they enclose is given by

$$\iint_R f(x,y) - g(x,y)\,\mathrm{d}A$$

where R is the region inside the curve of intersection of f and g. Note that $f(x, y) - g(x, y)$ gives us the height of the solid.

3. **Triple Integrals:** Again we defined triple integrals in a similar way to double and single integrals. We use iterated integrals to compute the triple integrals.

   (a) We integrate $f(x, y, z)$ over a box $B = [a, b] \times [c, d] \times [s, t]$ by using an iterated integral
   $$\iint_R f(x,y,z)\,\mathrm{d}V = \int_a^b \int_c^d \int_s^t f(x,y,z)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x.$$
   In this case, the order of integration can be interchanged without hesitation and we can write the above integral in five more different ways. Note that if $f(x, y, z) = g(x)h(y)k(z)$, then
   $$\iiint_B f(x,y,z)\,\mathrm{d}V = \left(\int_a^b g(x)\,\mathrm{d}x\right)\left(\int_c^d h(y)\,\mathrm{d}y\right)\left(\int_s^t k(z)\,\mathrm{d}z\right).$$

   (b) If a region has the form $B := \{(x, y, z)|a \le x \le b,\ g_1(x) \le y \le g_2(x),\ h_1(x, y) \le z \le h_2(x, y)\}$ then
   $$\iiint_B f(x,y,z)\,\mathrm{d}V = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x.$$
   The order is important! Similar formulas for the cases where we switch x, y, z.

Remark: In particular, we can find area of regions and volume of solids by using double and triple integral respectively.

$$\text{Area(R)} = \iint_R 1\,\mathrm{d}A \text{ and Volume(B) } = \iiint_B 1\,\mathrm{d}V.$$

4. **Change Of Variable I:** We learnt various necessary ways of simplifying our computation of integrals by introducing the change of variables. Below we summarize all of them.

   (a) **Polar:** If we take $x = r\cos(\theta),\ y = r\sin(\theta)$ then
   $$\iint_R f(x,y)\,\mathrm{d}A = \iint_{\text{New limits}} f(r\cos(\theta); r\sin(\theta))\ r\mathrm{d}r\mathrm{d}\theta.$$

(b) **Cylindrical:** If we take $x = r\cos(\theta),\ y = r\sin(\theta),\ z = z$ then

$$\iiint_B f(x,y,z)\,\mathrm{d}V = \iiint_{\text{New limits}} f(r\cos(\theta), r\sin(\theta), z)\ r\mathrm{d}r\mathrm{d}z\mathrm{d}\theta.$$

(c) **Spherical:** If we take $x = (\rho\sin(\phi))\cos(\theta),\ y = (\rho\sin(\phi))\sin(\theta),\ z = \rho\cos(\phi)$ then

$$\iiint_B f(x;y;z)\,\mathrm{d}V = \iiint f\left(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta)\,, \rho\cos(\phi)\right)\rho^2\sin(\phi)\,\mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta.$$

5. **Change Of Variable II:** In general, if we take $x = x(u,v), y = y(u,v)$ then we define the **Jacobian** of this change of variable as $\frac{\partial(x,y)}{\partial(u,v)} = det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$ and

$$\iint_R f(x,y)\,\mathrm{d}A = \iint_S f(x(u,v), y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\mathrm{d}u\mathrm{d}v$$

where S is the region in the new axis system u-v plane which is the inverse image of the region R in the above defined change of variable.

More generally, if we take $x = x(u,v,w), y = y(u,v,w), z = z(u,v,w)$ then we define the **Jacobian** of this change of variable as $\frac{\partial(x,y,z)}{\partial(u,v,w)} = det\begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}$ and

$$\iiint_E f(x,y)\,\mathrm{d}V = \iint_S f(x(u,v,w), y(u,v,w), z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\mathrm{d}u\mathrm{d}v\mathrm{d}w$$

where S is the region in the new axis system u-v-w space which is the inverse image of the region E in the above defined change of variable. This justifies the cylindrical and the spherical change of variables.

Tips:

(a) $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \left(\frac{\partial(u,v,w)}{\partial(x,y,z)}\right)^{-1}$. Same for two variables!

(b) When you do the change of variable in an integral, make sure that you completely get rid of the old variables (x, y, z).

(c) You are expected to be able to do basics examples: parallelograms, ellipses, ellipsoids, triangles, region between two curves.

Homework, Quiz and some extra problems all at one place

1. Find the maximum of $f(x, y) = x + y$ on the set where $x^4 + y^4 = 1$ and give the point where this occurs.

2. Find the points on the sphere $x^2 + y^2 + z^2 = 1$ that are closest and farthest from the point $(2, 1, 2)$.

3. Find the maximum and minimum values of $f(x, y, z) = x^2y^2z^2$ subject to the constraint $x^2 + y^2 + z^2 = 1$. (Hw)

4. Find the points on the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ where the tangent plane is parallel to the plane $3x - y + 3z = 1$. (Hw)

5. A package in the shape of a rectangular box can be mailed by the US Postal Service if the sum of its length and girth(the perimeter of a cross section perpendicular to the length) is 108 in. Find the dimensions of the package with largest volume that can be mailed. (Quiz)

6. A chemical company plans to construct an open (i.e., no top) rectangular metal tank to hold 256 cubic feet of liquid. It wants to determine the dimensions of the tank that will use the least amount of metal.

   (a) Set up the problem in the form to use the method of Lagrange multipliers, i.e., minimize F subject to the constraint G = c.

   (b) Determine the system of equations that has to be solved in order to solve the problem.

   (c) Solve the equations obtained in the above step.

7. Compute the integral of $f(x, y) = x^{2/3}y^{1/3}$ over the rectangle with vertices (0, 1), (2, 1), (2, 3) and (0, 3).

8. Find the volume of the region bounded by the graph of $z = x\sqrt{x^2 + y}$ and the planes $x = 0,\ x = 1,\ y = 0,\ y = 1$, and $z = 0$.

9. Evaluate $\iint_R xcos(y)\,\mathrm{d}A$ over the region R bounded by $y = 0,\ y = x^2$, and $x = 1$.

10. Let S_1 be the square with center vertices (1, 1), (1, -1), (-1, -1) and (-1, 1). Let S_2 be the square with vertices (0, 1), (1, 0), (-1, 0) and (0, -1). Let R be the region inside S_1 and outside S_2. compute $\iint_R x^2\,\mathrm{d}A$.

11. Evaluate the following iterated integrals.

(a) $\int_0^1 \int_x^{3x} 2ye^{x^3}\,\mathrm{d}y\,\mathrm{d}x$

(b) $\int_0^{\pi/2} \int_z^{\pi/2} \int_0^{sinz} 3x^2 siny\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

(c) $\int_2^4 \int_{-1}^1 (x^2 + y^2)\,\mathrm{d}y\,\mathrm{d}x$ (Hw)

(d) $\int_0^1 \int_1^2 \frac{xe^x}{y}\,\mathrm{d}y\,\mathrm{d}x$ (Hw)

12. Evaluate $\iint_D \frac{4y}{x^3+2}\,\mathrm{d}A,\ D = \{(x, y)|1 \le x \le 2,\ 0 \le y \le 2x\}$. (Hw)

13. Evaluate $\iint_D e^{y^2}\,\mathrm{d}A,\ D = \{(x, y)|0 \le y \le 1,\ 0 \le x \le y\}$.(Hw)

14. By interchanging the order of integration compute the following integrals:

   (a) $\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, \mathrm{d}y\, \mathrm{d}x$. (Hw)

   (b) $\int_0^1 \int_y^1 \cos(\frac{1}{2}\pi x^2)\, \mathrm{d}x\, \mathrm{d}y$. (DI-extra.pdf)

15. Find the area of the solid bounded by the paraboloid $z = x^2 + 3y^2$ and the planes $x = 0,\ y = 1,\ y = x$ and $z = 0$.

16. Find the area of the solid bounded by the cylinders $x^2 + y^2 = 4$ and $y^2 + z^2 = 4$.

17. Find the volume of the solid bounded by the elliptic paraboloid $z = 1 + (x-1)^2 + 4y^2$, the planes $x = 3$ and $y = 2$, and the coordinate planes. (Hw)

18. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and within the cylinder $x^2 + y^2 \leq 1,\ z \geq 0$. (DI-extra.pdf)

19. Let Ω be the region in xy-plane that is bounded below by $x-$axis and above by the line $y = 9$ and on the sides by $x = \sqrt{y}$, and the line $x = 3$.

(a) Express the integral $\iint_\Omega \sin(\pi x^3)\,\mathrm{d}A$ as a repeated integral, integrating first with respect to y.

(b) Express the integral $\iint_\Omega \sin(\pi x^3)\,\mathrm{d}A$ as a repeated integral, integrating first with respect to x.

(c) Choose one of the integral to evaluate the integral.

20. Let Ω be the triangular region formed by x-axis, $2y = x, x = 2$ and $f(x, y) = e^{x^2}$.

(a) Express the double integral $\iint_\Omega f(x, y)\,\mathrm{d}A$ as a repeated integral, integrating first with respect to y.

(b) Express the double integral $\iint_\Omega f(x, y)\,\mathrm{d}A$ as a repeated integral, integrating first with respect to x.

(c) Choose one of the integral to evaluate the integral.

21. Let Ω be the region $\{(x,y)|x^2+y^2 \le a^2,\ y \ge x\}$.

(a) Draw a sketch of the region and shade it.

(b) Express the double integral $\iint_\Omega f(x,y)\,\mathrm{d}A$ as a repeated integral, integrating first with respect to x.

(c) Express the double integral $\iint_\Omega f(x,y)\,\mathrm{d}A$ as a repeated integral, integrating first with respect to y.

(d) Express the region R in polar coordinates and use this description to evaluate $\iint_R \sqrt{a^2-y^2}\,\mathrm{d}A$.

22. Evaluate the following integral by converting it into polar coordinates:

(a) $\iint_R (x+y)\,\mathrm{d}A$, where R is the region that lies to the left of the y-axis between the circles $x^2+y^2=1$ and $x^2+y^2=4$. (Hw)

(b) $\iint_R ye^x\,\mathrm{d}A$, where R is the region in the first quadrant enclosed by the circle $x^2+y^2=25$. (Hw)

(c) $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, \mathrm{d}y\, \mathrm{d}x$. (Hw)

(d) $\int_{1/2}^1 \int_0^{\sqrt{1-x^2}} \mathrm{d}y\, \mathrm{d}x$. (DI-extra.pdf)

23. Use polar coordinates to find the volume of the following solids.

(a) The solid that is bounded above by the paraboloid $z = 1 + 2x^2 + 2y^2$ and the plane $z = 7$ in the first octant.(Hw)

(b) The solid that is bounded above by the paraboloid $z = 1 - (x^2 + y^2)$ and below by the paraboloid $z = x^2 + y^2$. (DI-extra.pdf)

24. Evaluate the repeated integral $\int_0^{1/2}\int_0^{\sqrt{1-x^2}} xy\sqrt{x^2+y^2}\,\mathrm{d}y\,\mathrm{d}x$ and also sketch the region determined by the limits of integration.

25. Evaluate the following triple integrals.

(a) $\int_0^1 \int_x^{2x} \int_0^y 2xyz \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x$. (Hw)

(b) $\iiint_E yz \cos(x^5) \, \mathrm{d}V$, where $R = \{(x, y, z) | 0 \leq x \leq 1, \ 0 \leq y \leq x, \ x \leq z \leq 2x\}$. (Hw)

(c) $\iiint_E xz \, \mathrm{d}V$, where E is the solid tetrahedron with vertices (0, 0, 0), (0, 1, 0), (1, 1, 0), and (0, 1, 1). (Hw) (You should know how to set up this integral in all six different ways)

26. Set up a triple integral to find the volume of the following solids.

(a) The solid bounded above by $z = 2x$ and below by the disc $(x-1)^2 + y^2 \leq 1$.

(b) The solid bounded by the cylinder $y = x^2$ and the planes $z = 0,\ z = 4$, and $y = 9$. (Hw)

27. Find the volume of the solid bounded below by the xy-plane and above by the spherical surface $x^2 + y^2 + z^2 = 4$ and on the sides by the cylinder $x^2 + y^2 = 4$.

28. Use Triple integral to evaluate the volume of the solid bounded above by the cylinder $y^2 + z = 4$, below the plane $y + z = 2$ and on the sides by $x = 0$ and $x = 2$. (DI-extra.pdf)

29. Find the volume of the solid bounded above by the plane $y + z = 2$, below by the $xy-$plane, and on the sides by $x = 6$ and $y = \sqrt{x}$. (DI-extra.pdf)

30. Find the volume of the solid bounded above by the hemisphere $z = \sqrt{4 - x^2 - y^2}$ and below by the cones $z = \sqrt{3(x^2 + y^2)}$. (DI-extra.pdf)

31. The cylinder $x^2 + y^2 = 4,\ z \geq 0$ is sliced by the plane $z = 4 + y$. Determine the volume of the "sliced" cylinder. (DI-extra.pdf)

32. Convert $\int_0^3 \int_0^{\sqrt{9-y^2}} \int_0^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$ into a triple integral in cylindrical coordinates. Sketch the solid determined by the limits. (DI-extra.pdf)

33. Integrate $f(x, y, z) = 2z$ over the solid S in the first octant bounded above by the paraboloid $z = 9 - x^2 - y^2$, below by the $xy-$plane, and on the sides by the planes $x = \sqrt{3}y$ and $y = \sqrt{3}x$. (DI-extra.pdf)

34. Set up a triple integral in cylindrical coordinates that gives the volume of the solid bounded above by $z = -1$, below by xy-plane and on the sides by the cylinders $y = x^2$ and $y = x$.

35. A solid S, in the first octant, is bounded above by the sphere $x^2 + y^2 + z^2 = 4$ below by the $xy-$plane, and on the sides by the planes $y = x$, $x = 0$, and the cylinder $x^2 + y^2 = 1$. Set up a triple integral in cylindrical coordinates that gives the volume of S.

36. Let V denote the triple integral $\int_0^3 \int_0^{6-x} \int_0^{2x} \mathrm{d}V$.

(a) Express the triple integral V as a repeated integral integrating in the order $dy\, dx\, dz$.

(b) Express the triple integral V as a repeated integral integrating in the order $dy\, dz\, dx$.

(c) Choose one of the integral to evaluate the integral.

37. Find the cylindrical coordinates of the point with rectangular coordinates $(1, 3\pi/2, 2)$. (Hw)

38. Find the spherical coordinates of the point with rectangular coordinates $(2, 2, \frac{2}{3}\sqrt{6})$.

39. Evaluate using triple integral $\iiint_T (x^2+y^2+z^2)dx\, dy\, dz$ where $T: 0 \le x \le \sqrt{4-y^2},\ 0 \le y \le 2, \sqrt{x^2+y^2} \le z \le \sqrt{4-x^2-y^2}$.

40. Let $a > 0$, and let E be the region enclosed between the spheres $x^2 + y^2 + z^2 = 2az$ and $x^2 + y^2 + z^2 = 3a^2$. Using a triple integral compute the volume of the region E.

41. Let E be the region bounded by $y = 0$, $y = 1 - x^2$, and $z = 1 - x^2$. Evaluate $\iiint_E xyz \, \mathrm{d}V$.

42. Use cylindrical coordinates to compute the integral $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$. (Hw)

43. Use spherical coordinates to compute the integral $\int_{-3}^{3}\int_{0}^{\sqrt{9-y^2}}\int_{0}^{9-x^2-y^2} z\sqrt{x^2+y^2+z^2}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$. (DI-extra.pdf)

44. Use spherical coordinates to compute the integral $\int_{-1/2}^{1/2} \int_{-\sqrt{1/4-x^2}}^{\sqrt{1/4-x^2}} \int_{\sqrt{3x^2+3y^2}}^{1-x^2-y^2} 1 \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x$. (DI-extra.pdf)

45. Sketch the region determined by an integral $\int_0^{\pi/4} \int_0^{2\pi} \int_0^{\sec\phi} \rho^2 \sin\phi \, \mathrm{d}\rho \, \mathrm{d}\theta \, \mathrm{d}\phi$. Also, interpret what this integral represents and then finally compute it. (DI-extra.pdf)

46. Evaluate $\iiint_{E_a} \frac{e^{-z}}{(x^2+y^2+1)^2}$, where E_a is the solid determined by the cylinder $x^2+y^2=a^2$ and the plane $z=0$ and $z=a$. Then finally give the value of $\lim_{a\to\infty} \iiint_{E_a} \frac{e^{-z}}{(x^2+y^2+1)^2}$. (Hw)

47. Find the Jacobian of the transformation $x = e^{u-v},\ y = e^{u+v},\ z = e^{u+v+w}$. (Hw)

48. Use appropriate transformation to compute the integral $\iint_R (4x + 8y)\,\mathrm{d}A$, where R is the parallelogram with vertices (-1, 3), (1, -3), (3, -1), and (1, 5). (Hw)

49. Evaluate $\iint_R \cos\left(\frac{y-x}{y+x}\right)\, \mathrm{d}A$, where R is the trapezoidal region with vertices (1, 0), (2, 0), (0, 2), and (0, 1).

50. Using a suitable change of variables compute $\iint_R \cos(9x^2 + 4y^2)\,\mathrm{d}A$; where R is the ellipse $9x^2 + 4y^2 \leq 1$.