DEPARTMENT OF MATHEMATICAL SCIENCES
CARNEGIE MELLON UNIVERSITY

Math 21-259 Calculus in 3D
Practice Final Exam

Allowed Time: 150 mins

1. (15 points) Find symmetric equations for the line of intersection L of the two planes $x + y + z = 1$ and $x - 2y + 3z = 1$. Also, find the angle between these two planes.

2. (15 points) Let $\mathbf{r}(t) = (\sqrt{2}t, e^t, e^{-t})$.

   (a) Calculate the arc length function $s(t)$ measured from t = 0.

   (b) Find the equation of the line tangent to the curve at the point $\mathbf{r}(1)$.

   (c) Compute the unit tangent vector $\hat{\mathbf{T}}(t)$.

   (d) Compute $\kappa(t)$.

3. (10 points) Find all local maximum, local minimum, and saddle points of $f(x, y) = e^{4y-x^2-y^2}$.

4. (20 points) Find the absolute maximum and minimum values of $f(x, y) = x^2 + y^2 + x^2y + 4$ on the set $D = \{(x, y) : -1 \leq x \leq 1,\ -1 \leq y \leq 1\}$. Also, give the points at which the function attains its maximum and minimum values.

5. (20 points) Find the dimensions of the rectangular box of maximum volume if the total surface area is given as 64 cm^2 using the method of Lagrange multipliers.

6. (15 points) Let $T = \{(x, y, z) : 0 \leq z \leq 6, z/2 \leq x \leq 3, x \leq y \leq 6 - y\}$ be the solid in space. Set up(not compute) a triple integral in the order $\mathrm{d}x\mathrm{d}y\mathrm{d}z$ that gives the volume of the solid T.

7. (15 points) Find the volume of the solid in the first octant which is bounded by the cone $x^2+y^2=3z^2$, by the planes $x=0$ and $x=\sqrt{3}y$, and by the sphere $4x^2+4y^2+4z^2=1$.

8. (20 points) Evaluate $\iint_R (x+y)\cos\pi(2x^2+xy-y^2)\,\mathrm{d}x\,\mathrm{d}y$ where R is the parallelogram with vertices (0, 0), (1, -1), (1/3, 2/3), and (4/3, -1/3).

9. (a) (15 points) Determine whether or not the vector filed $\mathbf{F}(x, y, z) =< yz, xz, xy >$ is conservative. If it is then f such that $\nabla f = \mathbf{F}$.

(b) (5 points) Compute $\int_C \mathbf{h}.\mathrm{d}\mathbf{r}$ where C is given by $\mathbf{r}(t) = (t\cos t)\mathbf{i} + (t^2\cos t + 3\sin^5(t))\mathbf{j}$, $0 \leq t \leq \pi/2$.

10. (20 points) Compute the line integral of the vector field $\mathbf{F}(x, y) =< xy, x^2y >$ over the boundary of the triangle with vertices (0, 0), (0, 1), (2, 1) directly **and** by using Green's theorem.

11. (10 points) Find the surface area of the part of the surface $z = 1 + 3x + 2y^2$ that lies above the triangle with vertices (0, 0), (0, 1), and (2, 1).

12. (20 points) Evaluate the surface integral $\iint_S \text{curl}\mathbf{F}.d\mathbf{S}$, where $\mathbf{F}(x, y, z) = -xy\mathbf{j} - xz\mathbf{k}$ and S consists of the paraboloid $z = x^2 + y^2,\ 0 \leq z \leq 1$, and the disk $x^2 + y^2 \leq 1,\ z = 1$ by following different ways:

(a) Directly.

(b) By using Stokes' Theorem.