Online Support
for Day 16 lecture

10.9 Motion of a particle in space

Position: $\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$

decribes the motion of the particle moving in space.

Velocity: $\vec{V}(t) = \dfrac{d\vec{r}}{dt} = \vec{r}\,'(t)$

— vector

Speed: $v(t) = |\vec{V}(t)| = |\vec{r}\,'(t)|$

— scalar

Acceleration: $\vec{a}(t) = \vec{r}\,''(t) = \dfrac{d\vec{V}}{dt}$

— vector

What do we mean by the derivative of the speed?

To answer this, let us suppose

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

$$|\vec{r}\,'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

$$\frac{d|\vec{r}\,'(t)|}{dt} \underset{\substack{\uparrow \\ \text{Chain} \\ \text{rule}}}{=} \frac{\left((x'(t))^2 + y'(t)^2 + z'(t)^2\right)'}{2\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}}$$

$$= \frac{2x'(t)x''(t) + 2y'(t)y''(t) + 2z'(t)z''(t)}{2\sqrt{x'(t)^2 + \cancel{2}y'(t)^2 + z'(t)^2}}$$

$$= \frac{x'(t)x''(t) + y'(t)y''(t) + z'(t)z''(t)}{\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}}$$

$$\frac{d|\vec{r}\,'(t)|}{dt} = \frac{\langle x'(t), y'(t), z'(t) \rangle \cdot \langle x''(t), y''(t), z''(t) \rangle}{|\vec{r}\,'(t)|}$$

$$= \frac{\vec{r}\,'(t) \cdot \vec{r}\,''(t)}{|\vec{r}\,'(t)|}$$

$$\boxed{\text{FORMULA: } \frac{d|\vec{r}\,'(t)|}{dt} = \frac{\vec{r}\,'(t) \cdot \vec{r}\,''(t)}{|\vec{r}\,'(t)|}}$$

$$\boxed{\text{FORMULA: } \frac{d|\vec{r}\,'(t)|}{dt} = \hat{T}(t) \cdot \vec{a}(t)}$$

Remark: Derivative of the speed is equal to the component of the acceleration along the unit tangent vector.

To be precise,

$$\boxed{\begin{aligned} |\vec{r}\,'(t)|' &= \operatorname{comp}_{\hat{T}} \vec{a} \\ &:= a_{\hat{T}} \quad \text{(Notation)} \end{aligned}}$$

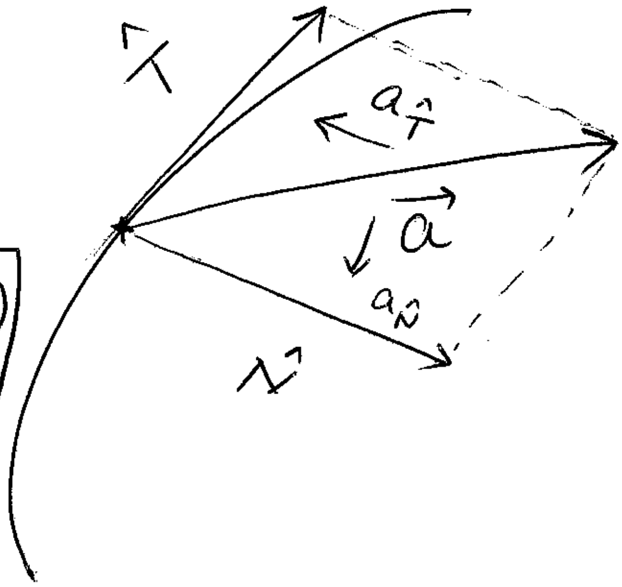
<u>Note</u>: $\dfrac{d|\vec{r}\,'(t)|}{dt} \neq \left| \dfrac{d\,\vec{r}\,'(t)}{dt} \right|$

In general, $\left| \dfrac{d\,\vec{r}(t)}{dt} \right| \neq \dfrac{d|\vec{r}(t)|}{dt}$

Component of acceleration

$$\text{Comp}_{\hat{T}} \vec{a} = a_{\hat{T}}$$
$$= |\vec{r}\,'(t)|'$$

$$\boxed{a_{\hat{T}} = \frac{\vec{r}\,'(t) \cdot \vec{r}\,''(t)}{|\vec{r}\,'(t)|}}$$



How to find
$$\text{Comp}_{\hat{N}} \vec{a} = a_{\hat{N}} \ ?$$

Consider $\quad \hat{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

$$\Rightarrow \quad \vec{r}\,'(t) = \hat{T}(t)\, |\vec{r}\,'(t)|$$
$$\Rightarrow \quad \vec{r}\,''(t) = \hat{T}'(t)\, |\vec{r}\,'(t)| + |\vec{r}\,'(t)|'\, \hat{T}(t)$$

$$\vec{a}(t) = \hat{T}'(t)\,|\vec{r}'(t)| + |\vec{r}'(t)|'\,\hat{T}(t)$$

$$\text{Comp}_{\hat{N}}\,\vec{a} = \vec{a}\cdot\hat{N}$$

$$= \left(\hat{T}'(t)\,|\vec{r}'(t)| + |\vec{r}'(t)|'\,\hat{T}(t)\right)\cdot\hat{N}(t)$$

$$= |\vec{r}'(t)|\,\hat{T}'(t)\cdot\hat{N}(t) + |\vec{r}'(t)|'\,\hat{T}(t)\cdot\hat{N}(t)$$

$$= |\vec{r}'(t)|\,\hat{T}'(t)\cdot\frac{\hat{T}'(t)}{|\hat{T}'(t)|}$$

$$= \frac{|\vec{r}'(t)|\;|\hat{T}'(t)|^2}{|\hat{T}'(t)|}$$

$$= |\vec{r}'(t)|\;|\hat{T}'(t)|$$

Recall, $\dfrac{|\hat{T}'(t)|}{|\vec{r}'(t)|} = \kappa(t)$

$$\text{Comp}_{\hat{N}} \vec{a} = |\vec{r}'(t)| \, \kappa(t) \, |\vec{r}'(t)|$$
$$= \kappa(t) \, |\vec{r}'(t)|^2$$
$$(\text{curvature} \cdot \text{speed}^2)$$

<u>Note</u>: Curvature $\uparrow$ - sharp turn
$\Longrightarrow$ $\uparrow \text{comp}_{\hat{N}} \vec{a}$ - Force exerted in the direction of normal.

High speed $\Rightarrow$ $\uparrow \text{comp}_{\hat{N}} \vec{a}$

$$\boxed{\text{Comp}_{\hat{N}} \vec{a} = \kappa(t) \, |\vec{r}'(t)|^2}$$

Recall theorem 10: $\kappa(t) = \dfrac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$

$$\boxed{\textbf{FORMULA:}\quad \mathrm{Comp}_{\hat{N}}\, \vec{a} = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)}}$$

SUMMARY

1. $\hat{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$

2. $\hat{N}(t) = \dfrac{\hat{T}'(t)}{|\hat{T}'(t)|}$

3. $\hat{B}(t) = \hat{T}(t) \times \hat{N}(t)$

4. $\kappa(t) = \left|\dfrac{\hat{T}'(t)}{|\vec{r}'(t)|}\right| = \dfrac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$

5. For plane curves $y = f(x)$,

$$\kappa(x) = \frac{|f''(x)|}{\left(1 + (f'(x))^2\right)^{3/2}}$$

6. $a_{\hat{T}} = |\vec{r}'(t)|' = \dfrac{\vec{r}'(t) \cdot \vec{r}''(t)}{|\vec{r}'(t)|}$

7. $a_{\hat{N}} = \dfrac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|}$