21123

Day 04
(Jan 17)

Last time

Thm: Every monotone, bounded sequence is convergent.

In fact, we were able to show (pictorially) that

$a_n \leq M$ and $a_n \uparrow \Longrightarrow$ convergent

$a_n \geq m$ and $a_n \downarrow \Rightarrow$ convergent.

what about the converse of the above theorem?

① Is every convergent sequence bounded?
Yes. We can prove it using (ε) epsilon definition

② Is every convergent sequence monotone?
No, if we take $a_n = \left(\frac{-1}{3}\right)^n$ then it is convergent but not monotone.

An idea behind constructing such an example is to multiply the convergent sequence by $(-1)^n$.

Mark T or F

1. Every bounded sequence is convergent.
   F, e.g. $a_n = (-1)^n$
2. Every monotone sequence is convergent
   F, e.g. $a_n = n$

Application of the above theorem.

Q1 Is the sequence $a_n = \left(1 + \frac{2}{n}\right)^n$ bounded?

Sol^n. $\lim_{n \to \infty} a_n = e^2$

Since $\{a_n\}$ is a convergent sequence therefore (by theorem) $\{a_n\}$ is bounded.

Q2 Is the sequence $a_n = \left(1 + \frac{2}{n}\right)^n$ monotonic?
Unfortunately, the above theorem cannot

help in this case. Use the knowledge of derivatives of functions to claim than a_n is an increasing sequence.

$$f(x) = \left(1 + \frac{2}{x}\right)^x, \quad x \geq 1$$

Use logarithmic differentiation!

Q3 Whate are bounds for a_n?

From above, a_n turns out to be $\uparrow$ which means $a_1 \leq a_n \leq \lim_{n \to \infty} a_n$

$$\boxed{3 \leq a_n \leq e^2}$$

One more interesting problem to think about: Take $b_n = (3^n + 4^n)^{1/n}$

ⓐ Is it bounded? If yes, then what are the bounds?

ⓑ Is it monotonic?

ⓒ Is it convergent? If yes, then what is the limit?

THINK before you go to your recitation tom.

Question from last time

Q If $a_n \xrightarrow{\text{diverges}} \pm\infty$ and $b_n \xrightarrow{\text{diverges}} \pm\infty$

then $a_n b_n \longrightarrow$?

Ans $a_n b_n$ also diverges and diverges to either $+\infty$ or $-\infty$ depending on the sign of product of the "limits".

It requires some proving which we are skipping in here!

Q what about $a_n + b_n$?

It is trickier!

If $a_n \xrightarrow{d} \infty$ and $b_n \xrightarrow{d} \infty$ then

$$a_n + b_n \xrightarrow{d} \infty$$

If $a_n \xrightarrow{d} -\infty$ and $b_n \xrightarrow{d} -\infty$ then

$$a_n + b_n \xrightarrow{d} -\infty$$

But we cannot say anything in other cases.

Q1 Given the following sequence:

$$a_1 = 1 \qquad a_n = \frac{1}{2}\left(a_{n-1} + \frac{R}{a_{n-1}}\right), \; n \geq 2$$

If the limit of a_n exists then find

$$\lim_{n\to\infty} a_n \; ?$$

Sol^n Let us denote $\lim\limits_{n\to\infty} a_n$ by L.

Note: If $\lim\limits_{n\to\infty} a_n = L$ then $\lim\limits_{n\to\infty} a_{n-1} = L$

The above note tells us that

$$\frac{1}{2}\left(a_{n-1} + \frac{R}{a_{n-1}}\right) \longrightarrow \frac{1}{2}\left(L + \frac{R}{L}\right)$$

This means that

$$\begin{pmatrix}\text{Limit of} \\ \text{L.H.S}\end{pmatrix} L = \frac{1}{2}\left(L + \frac{R}{L}\right) \begin{pmatrix}\text{limit of} \\ \text{R.H.S}\end{pmatrix}$$

solve for L: $L^2 = R \Rightarrow L = \pm\sqrt{R}$

How do we decide if $L = \sqrt{R}$ or $-\sqrt{R}$?
The given $\{a_n\}$ is a positive term sequence.
So, ~~the~~ $\lim\limits_{n\to\infty} a_n$ has to be positive

That is, $\boxed{L = \sqrt{R}}$.

<u>Note</u>: This sequence is neither increasing nor decreasing.

The sequence given by

$$a_1 = 1, \quad a_n = \frac{1}{2}\left(a_{n-1} + \frac{R}{a_{n-1}}\right) \text{ is}$$

used by Babylonians (predated Newton) to find decimal expansion of square roots.

<u>How?</u>

In particular, let us take $R = 3$.
To find decimal expansion of $\sqrt{3}$, list the elements of the sequence. We know that it converges to $\sqrt{3}$.

$a_1 = 1$
$a_2 = 2$
$a_3 = \frac{7}{4} = 1.75$
$a_4 = 1.73214$
$a_5 = 1.73205081$

This is what most calculators give as the value!

In fact, most calculators use this ~~algo~~ algorithm to return the values of square roots.

<u>Connection</u> of <u>this with Newton's Method</u>

Newton actually went a step ahead and formulated the above problem as finding root of the equation $x^2 - R = 0$.

This allowed Newton to generalize this method for most equations.

Pretty much any equation $f(x) = 0$ can be solved using Newton's Method.

$\left\{ \begin{array}{l} \text{As long as the function is differentiable} \\ \text{and just few other "nice" properties.} \end{array} \right.$

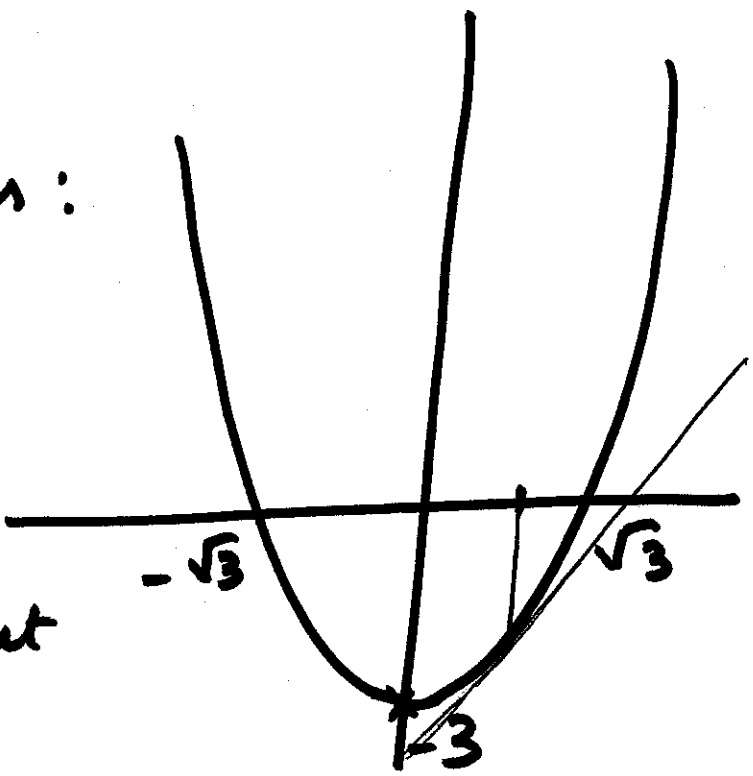
For instance, we don't have any formula that solves $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$

It would be nice to have an approximate solution (atleast) if exact solution is not plausible.

Newton's Method

We will illustrate the method using an example: $x^2 - 3 = 0$

1. Define f : $f(x) = x^2 - 3$.
2. Graph $f(x) = y$



3. Make an initial guess :
   $x_1 = 1$
   (why take $x_1 = 1$?)

We choose to stay close to the root and we know that it is at least bigger than 1.

4. Draw a tangent line at $(x_1, f(x_1))$
5. Find X - intercept of the tangent line at $(x_1, f(x_1))$: $y - f(x_1) = f'(x_1)(x - x_1)$

X - intercept : $x = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

Denote the x-intercept by x_2 and take that as the next guess.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1^2 - 3}{2x_1}$$

$$= \frac{x_1}{2} + \frac{3}{2x_1} = \frac{1}{2}\left(x_1 + \frac{3}{x_1}\right)$$

6. Repeat this process.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

($(n+1)^{th}$ iterate)

This, along with the initial guess, is called <u>Newton's sequence</u>.

<u>Next time</u>: For our example, sequence turns out to be

$$x_{n+1} = x_n - \frac{x_n^2 - 3}{2x_n}$$

$$= \frac{1}{2}\left(x_n + \frac{3}{x_n}\right) \text{: same as}$$

Babylonian sequence!