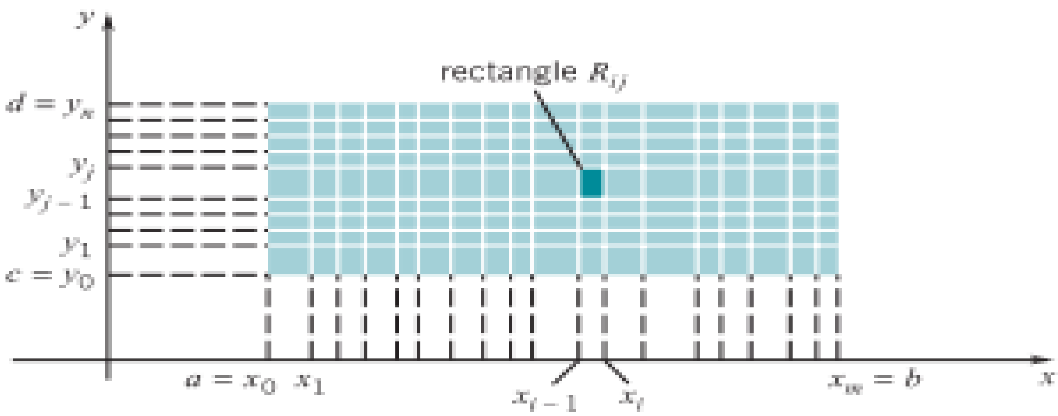
Summary of Chapter 12

Practice Problems

Chapter 12. Double and Triple Integrals

12.1 The Double Integral over a Rectangle

Let $f = f(x, y)$ be continuous on the Rectangle R: $a \le x \le b$, $c \le y \le d$.



The *double integral* of *f* over R= $\lim_{\Delta A_i \to 0} \sum f(x_i^*, \; y_i^*) \; \Delta A_i$ where $(x_i^*, \; y_i^*)$ is a sample point in R_{ij} .

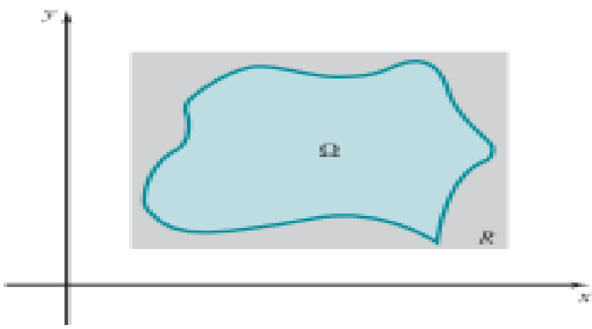
Notation: *double integral* of *f* over R= $I = \iint_{\Re} f(x, y) dxdy$

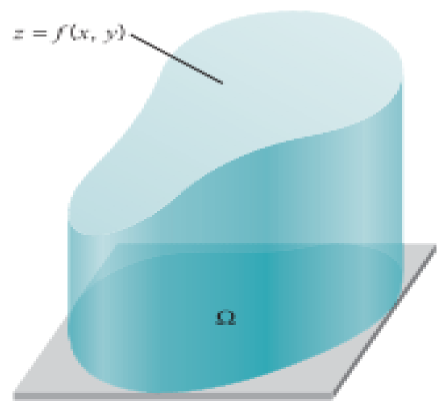
Note: Area element = dA = dx dy

Let Ω be an arbitrary closed bounded region in the plane. Then

$$\iint_{\Omega} f(x,y)dxdy = \iint_{\Re} F(x,y)dxdy$$

Where $\mathcal{R}$ is a rectangle that contains Ω, and $F(x, y) = f(x, y)$ on Ω and $F(x, y) = 0$ on $\mathcal{R} - \Omega$.





The double integral $\iint_{\Omega} f(x,y)dxdy = \iint_{\Re} F(x,y)dxdy$ gives the volume of the solid bounded below by Ω and above by $z = f(x, y)$.

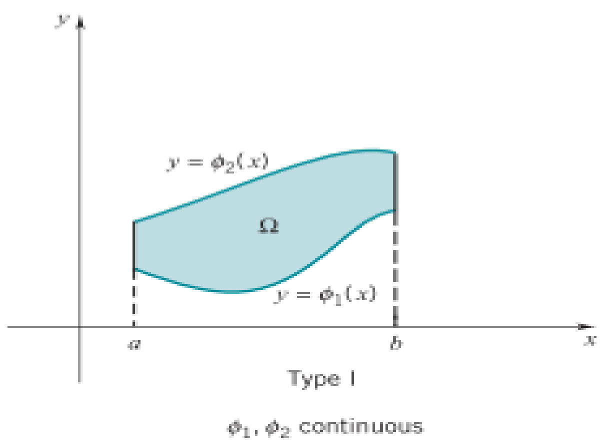
The physical meaning of the integral $\iint dxdy$ is the area of the region over which the integration is done.

Repeated/Iterated Integrals

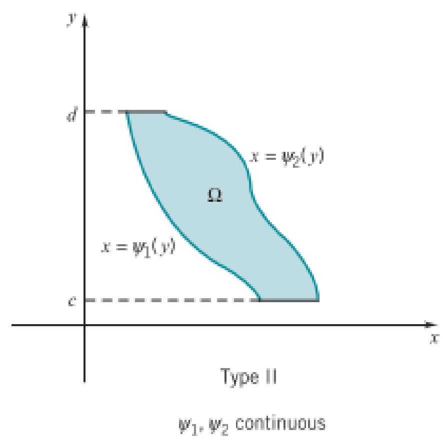
Complexity of double integrals comes from two sources:

1. **Function**
2. **Region**

Type I



Type II



Practice Examples

Example1. Find the area of the region enclosed by $y = x^2$ and $y = x^3$.

Example2. Evaluate the integral $\iint x^2 y$ over the region

$\Omega = \{(x, y): 0 \le x \le 4, 0 \le y \le 2\}$.

Example3. Evaluate the integral $\iint x^2 y$ over the region

$\Omega = \{(x, y): 0 \le x \le 4, 0 \le y \le x\}$.

Example4. Evaluate the integral $\iint \cos{(x + y)}$ over the region

$\Omega = \{(x, y): 0 \le x \le \pi/2, 1 \le y \le \pi/2\}$.

Example5. Evaluate the integral $\iint x + 3y^3 dxdy$ over the region

$\Omega = \{(x, y): 0 \le x^2 + y^2 \le 4\}$.

Example6. Find the volume under the paraboloid $z = x^2 + y^2$ within the cylinder $x^2 + y^2 \le 1, z \ge 0$.

Example7. Evaluate the integral $\iint 4 - y^2)$ over the region which is bounded between $y^2 = 2x$ and $y^2 = 8 - 2x$.

Example8. Sketch the region Ω that gives rise to the repeated integral and change the order of integration.

$$\int_0^1 \int_0^{y^2} f(x, y)$$

Example9. Evaluate $\int_0^1 \int_y^1 \cos \frac{1}{2} \pi x^2 \, dxdy$ by changing the order of integration.

Example10. Calculate by double integration the area of the bounded region determined by the given pair of curves.

$$y = x^2, \; x = 4y - y^2$$

Example11. Find the volume of the solid bounded by the coordinate planes and the plane $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1.$

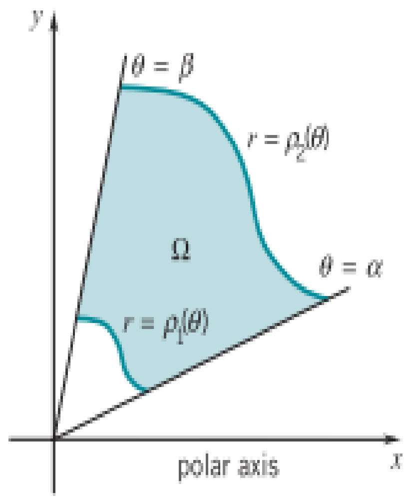
Example12. The given integral $\int_0^1 \int_{-y}^{y} f(x,y) dxdy$ is equal to

a. $2 \int_{-1}^{0} \int_{-x}^{1} f(x,y) dydx$
b. $2 \int_0^1 \int_x^1 f(x,y) dydx$
c. $\int_{-1}^{0} \int_{-x}^{1} f(x,y) dydx + \int_0^1 \int_x^1 f(x,y) dydx$
d. All of the above
e. None of the above

Example13. The given integral $\int_0^1 \int_{-y}^{y} sin\pi xy \, dxdy$ is equal to

a. $2 \int_0^1 \int_x^1 sin\pi xy \, dydx$
b. $\int_{-1}^{0} \int_{-x}^{1} sin\pi xy \, dydx + \int_0^1 \int_x^1 sin\pi xy \, dydx$
c. 0
d. All of the above
e. None of the above

12.3 **Double integrals and Polar coordinates**



How to find the area of the region Ω?

Single Integral: $\int_{\alpha}^{\beta} \rho_2^2(\theta)\, d\theta - \int_{\alpha}^{\beta} \rho_1^2(\theta)\, d\theta$

Double Integral: $\int_{\alpha}^{\beta} \int_{\rho_1(\theta)}^{\rho_2(\theta)} r\, drd\theta$

Area Element: $dA = rdr\, d\theta$

When to use polar coordinates for integration?

Watch out for Signal:

- Integrating over the unit disk or a part of the unit disk.

- Integrand involves $x^2 + y^2$

Practice Problems

Example1. Use Double integral to find the area of one leaf of the petal curve $r = 3sin3\theta$. (HINT: Sketch the curve in rectangular coordinates by plotting some angles. You will get a flower with three petals(leaves).)

Example2. Evaluate $\int_0^{\frac{\pi}{2}} \int_0^{3sin\theta} r^2 dr\, d\theta$

Example3. Calculate by changing to polar coordinates.

$$\int_{-1}^{1} \int_0^{\sqrt{(1-y^2)}} \sqrt{(x^2 + y^2)} dxdy$$

Example4. Use polar coordinates to evaluate the integral $\iint xy\, dxdy$ over unit disk.

Example5. Calculate by changing to polar coordinates.

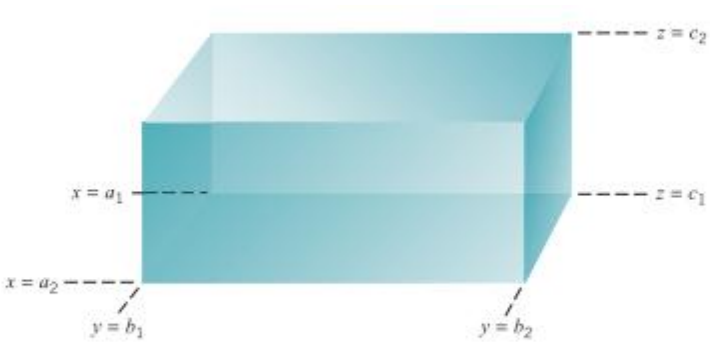
$$\int_{\frac{1}{2}}^{1} \int_0^{\sqrt{(1-x^2)}} dydx$$

Example6. Find the volume of the solid bounded below by the paraboloid $z = x^2 + y^2$ and above by the paraboloid $z = 1 - (x^2 + y^2)$.

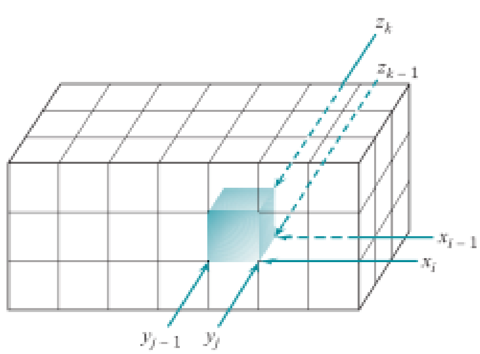
12.5 Triple Integrals

Take a function of three variables continuous on some portion T of three-space.

Integral over a box: $\Pi: a_1 \leq x \leq a_2, b_1 \leq y \leq b_2, c_1 \leq z \leq c_2$



Partition each edge of the box, B:

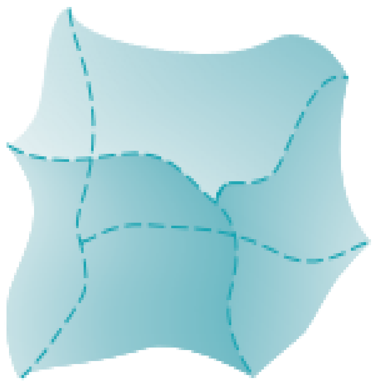


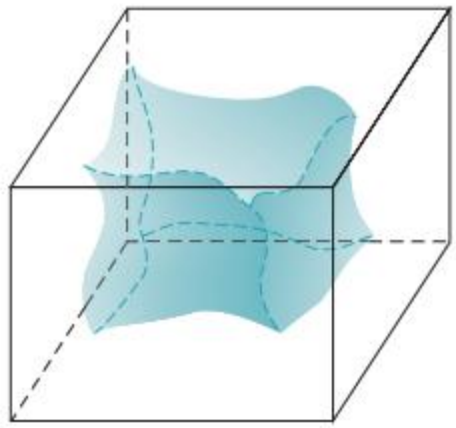
The *triple integral* of *f* over B= $\lim_{\Delta V_i \to 0} \sum f(x_i^*, y_i^*, z_i^*)\ \Delta V_i$ where (x_i^*, y_i^*, z_i^*) is a sample point in B_{ij} .

Notation: *Triple integral* of *f* over B=$\iiint_B f(x, y, z) dV$

Note: Volume element = dV = dx dy dz

Triple integral over a more general solid





$$\iiint\limits_{\Pi} f(x,y,z)dxdydz = \iiint\limits_{B} F(x,y,z)dxdydz$$

Where F(x, y, z) = f(x, y, z) for points on the domain Π $and\ F(x,y,z) =$ 0 on the points inside the box B but outside the domain Π.

Remember:

- Positivity: When $f(x, y, z) \geq 0$ then $\iiint_{\mathrm{T}} f(x, y, z) \geq 0$

- Physical Meaning: $\iiint_{\mathrm{T}} dxdydz =$ Volume of T

- Order: If $f(x, y) \geq g(x, y)$ then $\iiint_{T} f(x, y, z) dxdydz$ is greater than $\iiint_{\mathrm{T}} g(x, y, z)\, dxdydz$.

- Linearity: $\iiint_{\Pi}[af(x, y, z) + \ bg(x, y, z)]dxdydz$ $=a \iiint_{T} f(x, y, z) dxdydz +$ $b \iiint_{\mathrm{T}} g(x, y, z)\, dxdydz$

Repeated Triple Integrals

Reduction to a repeated integral

1. Type I: $a \le x \le b \quad \phi_1(x) \le y \le \phi_2(x) \quad \psi_1(x,y) \le z \le \psi_2(x,y)$
2. Type II: $c \le y \le d \quad \phi_1(y) \le x \le \phi_2(y) \quad \psi_1(x,y) \le z \le \psi_2(x,y)$
3. Type III:
4. ……
5. ……
6. ……

There are six possible types!

Practice Problems

Example1. Compare $\iiint_T dxdydz$ to $\iint_\Omega f(x,y)dxdy$ where $T = \{(x,y,z): (x,y) \in \Omega, 0 \le z \le f(x,y)\}$.

Example2. Evaluate $\int_0^1 \int_{1-x}^{1+x} \int_0^{xy} 4z dxdydz$.

Example3. Find the volume of the box where $2 \le x \le 4, 1 \le y \le 6, 0 \le z \le 2$ using triple integrals.

Example4. Calculate the triple integral $\iiint_T z dxdydz$ where T is the tetrahedron in the first octant bounded by the coordinate planes and the plane $x + y + z = 1$.

Example5. Find the volume of the solid bounded above by the cylindrical surface $x^2 + z = 4$, below by the plane $x + z = 2$, and on the sides by the planes $y = 0$ and $y = 3$.

Example6. Find the volume of the solid bounded above by the plane $y + z = 2$, below by the x, y-plane, and on the sides by $x = 6$ and $y = \sqrt{x}$.

Example7. Integrate $f(x,y,z) = 2z$ over the solid S in the first octant bounded above by the paraboloid $z = 9 - x^2 - y^2$, below by the xy-plane, and on the sides by the planes $x = \sqrt{3}\, y$ and $y = \sqrt{3}x$.

Example8. Find the volume of the solid bounded above by the hemisphere $z = \sqrt{4 - x^2 - y^2}$ and below by the cone $z = \sqrt{3(x^2 + y^2)}$.

Example9. Find the volume of the solid bounded above by the plane $y + z = 2$, below by the xy-plane, and on the sides by $x = 6$ and $y = \sqrt{x}$.

16. 8 Cylindrical coordinates

This coordinate system is used for a point P(x, y, z) in a space where polar is used for x, y coordinates and z is kept as it is.

$$(x, y, z) \to (r, \theta, z)$$

where

$$x = r\cos\theta \qquad r^2 = x^2 + y^2$$

$$y = r\sin\theta \qquad \tan\theta = \frac{y}{x}$$

$$z = z \qquad z = z$$

$$\iiint_{\Omega} f(x, y, z)dxdydz = \iiint_{\Gamma} F(r, \theta, z)rdrd\theta dz$$

where

$$F(r, \theta, z) = f(r\cos\theta, r\sin\theta, z)$$

Volume Element: $dV = rdr\, d\theta\, dz$

Watch out for same signals as for polar coordinates!

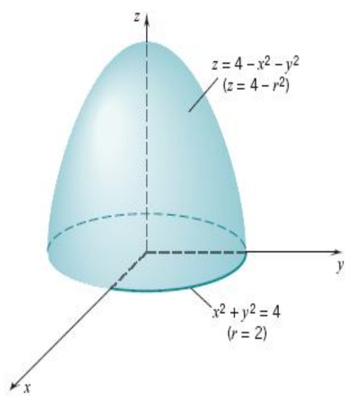
d

$=$

1. Given: $\int_0^{\pi} \int_0^{2} \int_0^{4-r^2} r\, dz\, dr\, d\theta.$

(a) Sketch the solid determined by the limits.

b) Evaluate the Integral:



c) Interpret the Result:

2. Convert

$$\int_0^{3} \int_0^{\sqrt{9-y^2}} \int_0^{\sqrt{9-x^2-y^2}} \sqrt{x^2+y^2}\, dz\, dx\, dy$$

into a triple integral in cylindrical coordinates. Sketch the solid determined by the limits.

3. The cylinder $x^2 + y^2 = 4$, $z \geq 0$ is sliced by the plane $z = 4 + y$. Determine the volume of the "sliced" cylinder.

4. Draw the solid that is bounded above by a portion of the hemisphere $z = \sqrt{1 - x^2 - y^2}$ and below by the cone $z = \sqrt{3x^2 + 3y^2}$.

Set up a triple integral that gives the volume of the solid and then find its volume.

Example5. Integrate $f(x, y, z) = 2z$ over the solid S bounded above by the paraboloid $z = 9 - x^2 - y^2$, below by the xy-plane, and on the sides by the planes $x = \sqrt{3}\, y$ and $y = \sqrt{3}x$. (in the first octant.)

Spherical Coordinates

This is another coordinate representation of a point P(x, y, z) in space.

$(x, y, z) \to (\rho, \theta, \phi)$ where

$$x = \rho \sin\phi \cos\theta \qquad \rho^2 = x^2 + y^2 + z^2$$

$$y = \rho \sin\phi \sin\theta \qquad \tan\theta = \frac{y}{x}$$

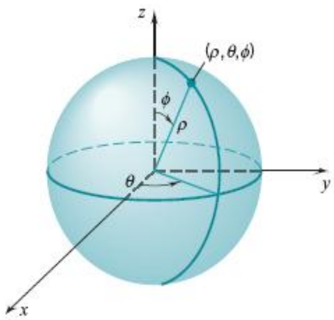
$$z = \rho \cos\phi \qquad \cos\phi = \frac{z}{\rho}$$

Triple integral:

$$\iiint_T f(x, y, z)\; dxdydz =$$

$$\iiint_S f(\rho \sin\phi \cos\theta,\ \rho \sin\phi \sin\theta,\ \rho \cos\phi)\, \rho^2 \sin\phi\; d\rho d\theta d\phi$$

What are they useful for? When are they useful?



Volume Element: $dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

v

1. Find the spherical coordinates of the points with rectangular coordinates $(2\sqrt{2}, -2\sqrt{2}, -4\sqrt{3}\ )$.

$=$

2. Find the rectangular coordinates of the point with spherical coordinates $(\frac{1}{\sqrt{2}}, \frac{\pi}{4}, \frac{1}{\sqrt{2}}\ )$.

3. Interpret $\rho = cos\phi$ geometrically.

1. Given

$$\int\int\int_{\Omega} \rho^3 \sin\phi \, d\rho \, d\phi \, d\theta =$$

$$\int_0^{2\pi} \int_0^{\pi/2} \int_0^{2} \rho^3 \sin\phi \, d\rho \, d\phi \, d\theta$$

(a) what is the integrand?

(b) What is Ω?

(c) Evaluate the integral.

2. Give the value of

$$\int_0^{\pi} \int_0^{\pi/2} \int_0^{3} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

without calculating the integral.

3. Evaluate

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-y^2}}\int_{0}^{\sqrt{9-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$$

4. Evaluate

$$\int_{-1/2}^{1/2}\int_{-\sqrt{\frac{1}{4}-x^2}}^{\sqrt{\frac{1}{4}-x^2}}\int_{\sqrt{3x^2+3y^2}}^{\sqrt{1-x^2-y^2}} 1\,dz\,dy\,dx$$

by changing to spherical coordinates.

5. The volume of a solid T is given by an integral in spherical coordinates. Sketch T and evaluate the integral.

$$\int_{0}^{\pi/4}\int_{0}^{2\pi}\int_{0}^{sec\phi} \rho^2 sin\phi\ d\rho d\theta d\phi$$