# MATH 21-259
# Summary of Chapter 13

1. **Vector Fields** are vector functions of two or three variables. Typically, a vector field is denoted by $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ where $P, Q, R$ are called scalar fields. The gradient of scalar function of two and three variables is really a vector field in plane or in space and is called a *gradient vector field.* A vector field is called *conservative* if it is the gradient of some scalar function, that is, $\mathbf{F} = \nabla f$ for some $f$. This function $f$ is called a *potential function* for $\mathbf{F}$.

2. **Line Integrals**

   (a) **Line Integral of scalar function:** If $f$ is a function of two variables and $C$ is a curve in plane parametrized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, a \leq t \leq b$, then the line integral of $f$ over $C$ is defined by

   $$\int_C f(x, y)\,\mathrm{d}s = \int_a^b f(x(t), y(t))|\mathbf{r}'(t)|\,\mathrm{d}t.$$

   If $f$ is a function of three variables and $C$ is a curve in space parametrized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, a \leq t \leq b$, then the line integral of $f$ over $C$ is defined by

   $$\int_C f(x, y, z)\,\mathrm{d}s = \int_a^b f(x(t), y(t), z(t))|\mathbf{r}'(t)|\,\mathrm{d}t.$$

   (b) **Line Integral of Vector Function:** If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a vector field of two variables and $C$ is a curve in plane parametrized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, a \leq t \leq b$, then the line integral of $\mathbf{F}$ over $C$ is defined by

   $$\int_C \mathbf{F}.\mathrm{d}\mathbf{r} = \int_C \mathbf{F}.\hat{\mathbf{T}}\,ds = \int_C P\mathrm{d}x + Q\mathrm{d}y = \int_a^b F(x(t), y(t)).\mathbf{r}'(t)\,\mathrm{d}t.$$

   If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field of three variables and $C$ is a curve in space parametrized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, a \leq t \leq b$, then the line integral of $\mathbf{F}$ over $C$ is defined by

   $$\int_C \mathbf{F}.\mathrm{d}\mathbf{r} = \int_C \mathbf{F}.\hat{\mathbf{T}}\,ds = \int_C P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z = \int_a^b F(x(t), y(t), z(t)).\mathbf{r}'(t)\,\mathrm{d}t.$$

   If $\mathbf{F}$ represents the force field then the line integral of $\mathbf{F}$ along a curve $C$ represents the work done by the force field $\mathbf{F}$ in moving a particle from the starting point of the curve to the final point of the curve.

3. **Fundamental Theorem of Line Integral** states that if $f$ is a differentiable of function of two or three variables and $C$ is the smooth curve given by the vector function $\mathbf{r}(t), a \leq t \leq b$, then

   $$\int_C \nabla f.\mathrm{d}\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

**Consequences/Remarks:**

(a) The integral of the gradient vector field along a smooth closed curve is always zero. In mathematical notation, $\oint_C \nabla f.\mathrm{d}\mathbf{r} = 0$.

(b) If $\mathbf{F}$ is a vector field defined everywhere then

$$\mathbf{F} = \nabla f \Longleftrightarrow \int_C \mathbf{F}.\mathrm{d}\mathbf{r} \text{ is path independent} \Longleftrightarrow \oint_C \mathbf{F}.\mathrm{d}\mathbf{r} = 0$$

for every closed curve $C$.

(c) If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a vector field defined everywhere then

$$\mathbf{F} = \nabla f \Longleftrightarrow Q_x = P_y.$$

To find $f$, we solve the equations $f_x = P$ and $f_y = Q$ simultaneously.

(d) If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field defined everywhere then

$$\mathbf{F} = \nabla f \Longleftrightarrow \ Q_x = P_y, \ Q_z = R_y, \text{ and } P_z = R_x \ \Longleftrightarrow \text{curl}\mathbf{F} = \mathbf{0}$$

where curl $\mathbf{F}$ is defined below. To find $f$, we solve the equations $f_x = P, \ f_y = Q$, and $f_z = R$ simultaneously.

4. **How to find $f$ so that $\mathbf{F}(x, y, z) = \nabla f(x, y, z)$?**

(a) **Step I.** To find $f$ we need to solve the following three equations:

$$\begin{aligned} f_x &= P && (1) \\ f_y &= Q && (2) \\ f_z &= R && (3) \\ & && (4) \end{aligned}$$

(b) **Step II.** Choose one of the equation from above and undo the operation of partial derivative by doing partial integration. Ideally, we choose the one that is easy to integrate. If for instance we choose (1) then remember the constant of integration would depend on $y$ and $z$. Thus, we get $f(x, y, z) = \int P \, \mathrm{d}x + g(y, z)$.

(c) **Step III.** We may now target on using other two equations to find the unknown function $g(y, z)$. We may start by equating the partial derivative of $f$ with respect to $y$ that we get from Step II and equation(2). This step would guarantee that we find the value of $g(y, z)$ in terms of known function of $y$ and an unknown function of $z$ which we may we call as $h(z)$. After plugging this information back into the original function, we see that only unknown we have now is $h(z)$.

(d) **Step IV.** Last step will focus on finding $h(z)$ by equating the partial derivative of $f$ that we obtain from Step III and equation 3. Finally, plug the value of $h$ into the function that we obtained to Step III to give the answer.

5. **Green's theorem** states that if $C$ is a piecewise smooth closed curve oriented counter-clockwise enclosing a region $D$ and if $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ where P and Q have continuous partial derivative on an open region that contains $D$, then

$$\oint_C \mathbf{F}.\mathrm{d}\mathbf{r} = \oint_C P\mathrm{d}x + Q\mathrm{d}y = \iint_D (Q_x - P_y)\,\mathrm{d}A.$$

Warning: Only true for closed curves!

**Application of Green's theorem**

(a) To compute the line integrals:

(b) To compute the area of the regions: Note that Area(D) $= \iint_D 1\,\mathrm{d}A = \oint_C \mathbf{F}.\mathbf{dr}$ where C is the boundary of the domain $D$ and $\mathbf{F}$ can be chosen as any of the following: $< 0, x >$ or $< -y, 0 >$ or $< -y/2, x/2 >$. It is generally advised to use $\mathbf{F} =< -y/2, x/2 >$ when dealing with circular curves and other two for other types of curves.

6. How to compute line integral?

(a) **Step I**. First of all, check if the given vector field $\mathbf{F}$ is conservative or not.

(b) **Step II**. If the answer to the above turns out to be YES (Feeling Lucky) then you may use fundamental theorem to compute the line integral provided finding a potential function is not that difficult. But if the answer is NO then you should ask another question, that is, Is the given curve closed?

(c) **Step III**. If the curve turns out to be closed then it is generally a good idea to use Green's theorem and compute the double integral instead of the line integral. But if the given curve is not closed then you may have to continue with the direct computation of the line integral.

All of the above steps can be summarized in the form of the following chart:

$$\begin{array}{ccccc}
 & & \text{Is } \mathbf{F} = \nabla f? & & \\
 & \swarrow & & \searrow & \\
 & \text{Yes} & & & \text{No, Is C a closed curve?} \\
 \swarrow & & & \swarrow & \qquad \searrow \\
\text{Use Fundamental Theorem of line integral} & & & \text{Yes} & \qquad \text{No} \\
 & & \swarrow & & \qquad \searrow \\
 & & \text{Use Green's theorem} & & \qquad \text{Direct computation}
\end{array}$$

7. **Curl and Divergence:** Let $\mathbf{F} =< P, Q, R >$ be a vector function then

$$\text{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = (R_y - Q_z)\mathbf{i} - (R_x - P_z)\mathbf{j} + (Q_x - P_y)\mathbf{k} \text{ and div}\mathbf{F} = P_x + Q_y + R_z.$$

8. **Surface Area:** Given a parametrized surface $S : \mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$ where $(u,v)$ belongs to domain $D$, then the surface area of this surface is given by

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, \mathrm{d}A,$$

where $\mathbf{r}_u$ and $\mathbf{r}_v$ are partial derivatives of $\mathbf{r}$ with respect to $u$ and $v$ respectively.

9. **Surface Integral**

   (a) **Surface Integral of scalar functions:** Given a scalar field $f(x,y,z)$ and a surface $S : \mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$ where $(u,v)$ belongs to domain $D$, then the surface integral of $f$ is defined by

   $$\iint_S f(x,y,z) \, \mathrm{d}S = \iint_D f(\mathbf{r}(u,v)) |\mathbf{r}_u \times \mathbf{r}_v| \, \mathrm{d}A.$$

   In particular, a surface $z = g(x,y), (x,y)$ in $D$ can be parametrized as follows: $\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + g(x,y)\mathbf{k}$, $(x,y)$ in $D$. Note that a normal vector to the tangent plane to this surface is given by $\mathbf{r}_x \times \mathbf{r}_y = -g_x\mathbf{i} - g_y\mathbf{j} + \mathbf{k}$. Thus the surface area of such a surface is given by

   $$\iint_D \sqrt{(g_x)^2 + (g_y)^2 + 1} \, \mathrm{d}A.$$

   (b) **Surface Integral of vector functions:** Given a scalar field $\mathbf{F}(x,y,z) =< P, Q, R >$ and a surface $S : \mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$ where $(u,v)$ belongs to domain $D$, then the surface integral of $\mathbf{F}$ is defined by

   $$\iint_S \mathbf{F}.\mathrm{d}\mathbf{S} = \iint_S \mathbf{F}(\mathbf{r}(u,v)).\hat{\mathbf{n}} \, \mathrm{d}S = \iint_S \mathbf{F}(\mathbf{r}(u,v)).\mathbf{r}_u \times \mathbf{r}_v \, \mathrm{d}A.$$

   In particular, if $z = g(x,y), (x,y)$ in $D$ is the given surface then

   $$\iint_S \mathbf{F}.\mathrm{d}\mathbf{S} = \iint_D -Pg_x - Qg_y + Rg_z \, \mathrm{d}A.$$

**Remarks:**

1. Note that a vector $\mathbf{r}_x \times \mathrm{r}_y$ above points in the upward direction as indicated by the position component of $\mathbf{k}$.
2. We say that the surface is **positively oriented** if the normal vector is pointing upwards.
3. As a convention, for a closed surface we take the positive orientation to be the one when the normal vector point outwards.
4. Closed surfaces have no boundary, e.g., sphere has no boundary.

10. **Stokes' Theorem** states that if $S$ is a positively oriented surface that is bounded by a closed curve C with positive orientation and $\mathbf{F} =< P, Q, R >$ where $P, Q$, and $R$ have continuous first order partial derivatives. Then

$$\int_C \mathbf{F}.\mathrm{d}\mathbf{r} = \iint_S \mathrm{curl}(\mathbf{F}).\mathrm{d}\mathbf{S}.$$

11. **Divergence Theorem** states that if $E$ is a simple solid region which has $S$ as positively oriented boundary surface and $\mathbf{F} =< P, Q, R >$ where $P, Q$, and $R$ have continuous first order partial derivatives. Then

$$\iint_S \mathbf{F}.\mathrm{d}\mathbf{S} = \iiint_E \mathrm{div}(\mathbf{F})\,\mathrm{d}\mathbf{V}.$$