Absolute Max/Min Problems

Q Find the absolute max and min of the function $f(x,y) = x^2+2y^2$ on D where D is the closed annular region centered at (0,0) of two radii 1 and 4

Sol

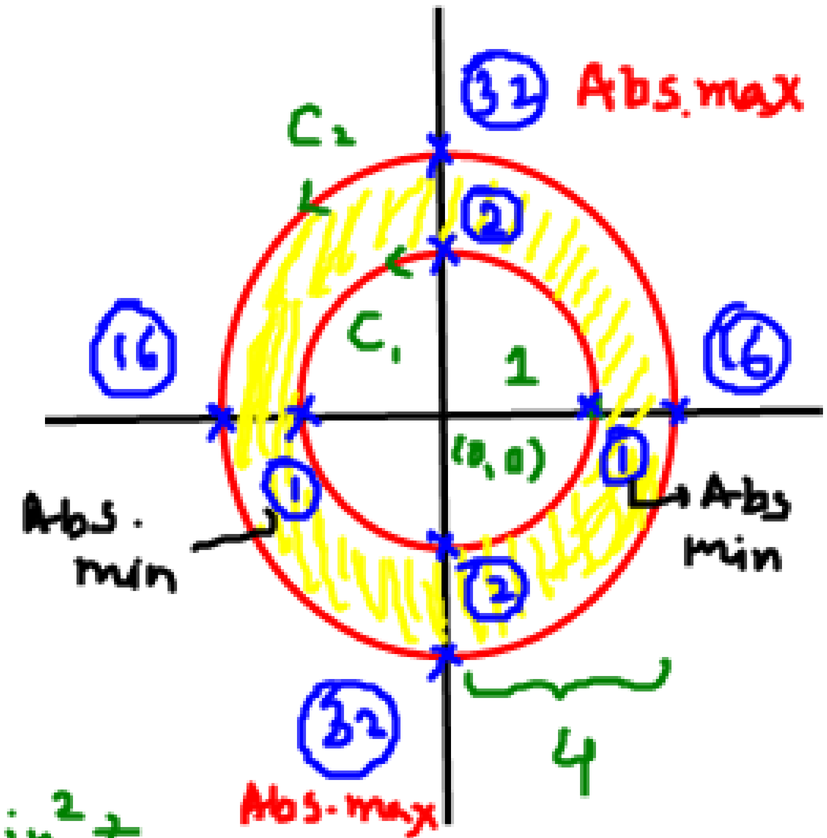
Soln Bdry consists of C_1 and C_2.

I $\nabla f = \langle 2x, 4y \rangle = 0$

$\Rightarrow x = 0 = y$

(0,0) is the critical point but it does not belong to the given domain. Ignore it!

II Find the extreme values of f on the bdry of the domain D.



On C_1:

$$x(t) = \cos t,\quad y(t) = \sin t,\quad 0 \le t \le 2\pi$$

$$f_1(t) = f(\cos t, \sin t)$$
$$= \cos^2 t + 2\sin^2 t$$
$$= \cos^2 t + \sin^2 t + \sin^2 t$$
$$= 1 + \sin^2 t \quad \text{(Use identity).}$$

$$f_1'(t) = 0 \Rightarrow 2\sin t\cos t = 0 \Rightarrow \sin 2t = 0$$
$$\Rightarrow 2t = 0, \pi, 2\pi, 3\pi, 4\pi$$

$t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \cancel{\frac{5\pi}{2}}$

end points

bigger than 2π.

$f_1(0) = f(1,0) = 1$

$f_1\left(\frac{\pi}{2}\right) = f(0,1) = 2$

Go back and label

$f_1(\pi) = f(-1,0) = 1$

same points.

$f_1\left(\frac{3\pi}{2}\right) = f(0,-1) = 2$

$f_1(2\pi) = f(1,0) = 1$

On C_2: $x(t) = 4\cos t$ $\quad 0 \le t \le 2\pi$

$y(t) = 4\sin t$

$f_2(t) = (4\cos t)^2 + 2(4\sin t)^2 = 4^2(\cos^2 t + 2\sin^2 t)$

Note that: $f_2(t) = 4^2 f_1(t)$

$\Rightarrow$ same critical points since constant does not matter.

(If you do not feel convinced then you should probably do the calculation one more time.)

$$f_2(0) = 4^2 f_1(0) = 16$$

$$f_2(\pi/2) = 4^2 f_1(\pi/2) = 32$$

$$f_2(\pi) = 4^2 f_1(\pi) = 16$$

$$f_2(3\pi/2) = 4^2 f_1(3\pi/2) = 32$$

$$f_2(2\pi) = 4^2 f_1(2\pi) = 16$$

Go back and label →

Conclusion:

Absolute maximum value $= 32$ at $(0, 4)$ and $(0, -4)$.

Absolute minimum value $= 1$ at $(1, 0)$ and $(-1, 0)$.