MATH 21-123

List of Important Ideas and Definition from Section 8.4-8.8

1. **Alternating Series Estimation Theorem**: If $s = \sum(-1)^{n-1}b_n$ is the sum of an alternating series that satisfies

   (i) $0 \leq b_{n+1} \leq b_n$ and

   (ii) $\lim_{n\to\infty} b_n = 0$

   then
   $$|R_n| \leq b_{n+1}$$
   The above theorem is useful is estimating the sum of an alternating series. For instance, we can say that the sum of the series $\sum \frac{(-1)^n}{n!}$ is equal to 3.68 to within the error less than 0.0002 (See Example 4 on pg 440).

2. **Absolutely convergent Series**: A series $\sum a_n$ is called absolutely convergent if the series $\sum |a_n|$ is convergent. An important fact about absolutely convergent series is that they are always convergent, that is, $\sum |a_n|$ converges $\Rightarrow \sum a_n$ converges.

3. **Conditionally convergent Series**: A series $\sum a_n$ is called conditionally convergent if it is convergent but not absolutely convergent.

4. **Power Series**: A general form of a power series centered at c is given by $\sum a_n(x-c)^n$.

5. **Power Series Theorem**: There are only three possibilities for any power series $\sum a_n(x-c)^n$:

   (a) It converges only at at the center c

   (b) It converges for all x

   (c) There is a positive number R such that the power series converges if $|x-c| < R$ and diverges if $|x-c| > R$.

6. **Consequences of Power Series Theorem**

   (a) Radius of Convergence: The number R in the Power series theorem is called the radius of convergence. To find R, we use either the Ratio test or the Root test.

   (b) Interval of Convergence: The maximal interval on which a power series converges. It could be anything out of four choices:
   $$(c-R,\ c+R),\ [c-R,\ c+R),\ (c-R,\ c+R],\ [c-R,\ c+R]$$
   In general, behavior at c - R and c + R is not predictable. Thus, checking at end points is important.

(c) If $\sum a_k x^k$ converges at $c \neq 0$ then it converges for all $|x| < |c|$ and if $\sum a_k x^k$ diverges at d then it diverges for all x such that $|x| > |d|$.

7. **Differentiation of Power Series**: This is also called term by term-by-term differentiation.

$$(\sum_{n=0}^{\infty} a_n (x-c)^n)' = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-c)^n$$

Note that R.O.C of $\sum_{n=0}^{\infty} a_n(x-c)^n$ = R.O.C of $\sum_{n=1}^{\infty} na_n(x-c)^{n-1}$, that is, f and f' have the same radius of convergence given that f represents a power series.

8. **Integration of Power Series**: This is also called term by term-by-term integration.

$$\int \sum_{n=0}^{\infty} a_n (x-c)^n \, dx = \sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1} + C = \sum_{n=1}^{\infty} a_{n-1} \frac{(x-c)^n}{n} + C$$

where C is the constant of integration. Note that R.O.C of $\sum_{n=0}^{\infty} a_n(x-c)^n$ = R.O.C of $\sum_{n=0}^{\infty} a_n \frac{(x-c)^{n+1}}{n+1} + C$, that is, f and $\int f(x)\, dx$ have the same radius of convergence given that f represents a power series.

9. **Representing Function as a Power Series**: To represent a function as a power series, we need the following tools:

   (a) Geometric series: $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ whenever $|x| < 1$.

   (b) Differentiation of Power Series

   (c) Integration of Power Series

10. **Taylor Series of f at c**: $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$.

11. **Maclaurin Series of f**: This is a special name given to the Taylor series of f centered at zero. It is given by $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$.

12. **Lagrange's form of Remainder**: For a given function f, the Lagrange's form of remainder is given by

$$R_n(x) = \frac{f^{(n+1)(z)}}{(n+1)!}(x-c)^{n+1}$$

where z is a number between c and x. Also, note that z depends on n. Sometimes, it is useful to deal with the remainder formula as follows:

$$|R_n(x)| \leq \max_z |f^{(n+1)}(z)| \frac{|x-c|^{n+1}}{(n+1)!}.$$

For instance, when we know that $|f^{(n)}(x)| < K$ for all x and n then the above form saves us from the trouble of worrying about z. For example, we can do this for $\cos x$ and $\sin x$.

13. **Taylor's Polynomial**: The n^{th} degree Taylor polynomial for a function f centered at c is given

$$T_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

14. **Representing Function as a Taylor Series**: There are two ways we represent a function f as Taylor Series.

    (a) If we know beforehand that f has a power series representation, that is, $f =$ power series centered at c for all x with $|x-c| < R$ then $f =$ sum of its Taylor Series centered at c for all x with $|x-c| < R$.

    (b) If we do not know that f has a power series representation then this method turns out to be quite effective. According to this method, if $\lim_{n\to\infty} R_n(x) = 0$ for all x with $|x-c| < R$ then $f =$ sum of its Taylor Series centered at c for all x with $|x-c| < R$.

    This method works in all situations but we prefer the power series expansion whenever possible as the power series expansion method is usually much faster and easier to apply.

    Remember that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$. This turns to be the key result in showing that $\lim_{n\to\infty} R_n(x) = 0$.

15. **Application of Series**: There are many applications of the Taylor series expansion.

    (a) To find the integral of a function with complicated antiderivative.
    (b) To find an approximation of a definite integral
    (c) To find the limit of certain function
    (d) To find a polynomial approximation of a function: In particular, there are two ways of finding an approximation for functions with alternating Taylor series — one uses alternating series estimation theorem and the other using the remainder form.
    (e) To solve differential equations

16. Following are the power series representation of some of the important function.

    1. $\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all x.
    2. $\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$ for all x.
    3. $e^x = \sum_{n=0}^{\infty}(-1)^n \frac{x^n}{n!}$ for all x.
    4. $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1} \frac{x^n}{n}$ for all $-1 < x < 1$.
    5. $tan^{-1}x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$ for all $-1 < x < 1$.
    6. $(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$ for all $-1 < x < 1$.
    7. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for all $-1 < x < 1$.