MATH 21-123

Tips on Using Tests of Convergence

1. **Geometric Series Test (GST)**: The use of this test is straightforward. You only use this when the series is in the form $\sum_{n=0}^{\infty} ar^n$. We say that the series converges if and only if $|r| < 1$ and the sum is given by, $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$, where a is the first term and r is the common ratio. Also note that $\sum_{n=0}^{k} ar^n = \frac{a(1-r^{k+1})}{1-r}$. To understand this formula, look for its proof into your notes. Note that terms of the series could even be negative.

2. **p-Series Test**: The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. This test can be proved using integral test(which is described next). You will probably never get to use this test individually in a problem but nevertheless this test has important role to play as it supports the working of other tests. We generally use this test along with comparison and limit comparison test.

3. **Integral Test**: This test states that if $a_n = f(n)$ for some function that is nonnegative and a continuous decreasing (eventually decreasing) function on $[1, \infty]$ then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\int_1^{\infty} f(x)\ dx$ converges. Make sure to check all the properties before you apply this test. In general, **you use this test when the terms have the configuration of a derivative**. For example: $\sum \frac{\ln n}{n}$, $\sum \frac{1}{n \ln n^p}$, $\sum ne^{-n}$ and etc. The convergence or the divergence for some of the above mentioned examples may also be shown using other tests, for instance, we could use comparison test for the first one, Root test or the Ratio test for the third one.

4. **Comparison Test (CT)**: First of all, note that this test is only valid for positive term series (like the integral test). This test states that if $\sum a_n$ is a series with **positive terms** and

   * $a_n \leq b_n$ such that $\sum b_n$ converges then $\sum a_n$ also converges.
   * $a_n \geq b_n$ such that $\sum b_n$ diverges then $\sum a_n$ also diverges.

   Typically, you choose b_n to be either the p-series or the geometric series(basically the series about which you know quite a lot in terms of its convergence). **Please be warned about the right inequalities**. It is easy to fall into the trap of justifying the wrong inequality.

5. **Limit Comparison Test (LCT)**: This test is also valid for positive term series ($\sum a_n,\ a_n \geq 0$) only. Again you choose b_n as in the case of the comparison test. The only difference is(which makes this test easy to use) that instead of checking for inequalities, you look for the limit of $\frac{a_n}{b_n}$.

   Formally speaking, this states that if $\sum a_n$ is a series with positive terms and you choose $\sum b_n$ such that $\lim_{n\to\infty} \frac{a_n}{b_n}$ is non-zero and finite then $\sum a_n$ converges(diverges) if and only if $\sum b_n$ converges(diverges).

Typically, you use this test when you are a given a series with n^{th} term given by a **rational function in term of** n. This by no means imply that you cannot use this test for series other the ones that are given by rational functions, for example, this test works best for $\sum \sin \frac{\pi}{n^2}$. Keep in mind that this **test fails if the limit turns out to be zero or infinity**. In this case, you should look for other test. It is possible that the same series might converge or diverge using some other test.

8. **Divergence test (DT)**: This test states that if $\lim_{n\to\infty} a_n$ either does not converge to zero or does not exist then the series $\sum a_n$ must diverge. The converse of this is not true. Other than checking the divergence of the series, this test also allows us find the limit of sequences. If the series $\sum a_n$ converges then $\lim_{n\to\infty} a_n = 0$. This is quite useful. For instance, if you recall we found $\lim_{n\to\infty} \frac{n!}{n^n} = 0$ using squeeze theorem. Another way to see this, show that the series $\sum \frac{n!}{n^n}$ converges which can be done using ratio test.

7. **Alternating Series Test (AST)**: This test is used for checking the convergence of an alternating series. As opposed to the divergence test, this test allows us to claim the convergence of the series $\sum(-1)^n a_n$ if in addition to $\lim a_n = 0$ we have that all the terms a_n are positive and decreasing.

   Formally speaking, if $\sum(-1)^n a_n$ is a series with $a_n \geq 0$ for all n such that

   * a_n is a decreasing sequence(eventually decreasing).
   * $\lim_{n\to 0} a_n = 0$

   then $\sum(-1)^n a_n$ converges.

   The converse of the above is not true. In other words, the above test is inconclusive if a_n's fail to decrease. Note that if $\lim a_n \neq 0$ then $\lim(-1)^n a_n$ does not exist which implies that $\sum(-1)^n a_n$ fail to diverge by divergence test.

7. **Ratio Test**: This test is very useful in checking for the absolute convergence of the series. Keep in mind that the absolute convergence for positive term series is same as convergence which means that this can be used for any kind of series.

   This test states that given any series $\sum a_n$.

   – If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$ then $\sum a_n$ converges absolutely.
   – If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| > 1$ then the given series $\sum a_n$ diverges.
   – If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$ then nothing can be said about the series. In other words, we say that the ratio is inconclusive.

NOTE: Always take the absolute value of the expression $\frac{a_{n+1}}{a_n}$.

This test is particularly effective with **factorials and with combination of powers and factorials**. If the terms are rational functions of n, the ratio test is inconclusive. For example: we know that the series $sum\frac{1}{n+1}$ diverges by LCT(say) but the ratio test turns out to be inconclusive.

Indeed, if we take $a_n = \frac{1}{n+1}$ then

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{n+1}{n+2} = 1$$

which implies that the ratio test is inconclusive.

8. **Root Test**: This test is very useful in checking for the absolute convergence of the series just like the ratio test.

   This test states that given any series $\sum a_n$.

   - If $\lim_{n\to\infty} |a_n|^{1/n} < 1$ then $\sum a_n$ converges absolutely.
   - If $\lim_{n\to\infty} |a_n|^{1/n} > 1$ then the given series $\sum a_n$ diverges.
   - If $\lim_{n\to\infty} |a_n|^{1/n} = 1$ then nothing can be said about the series. In other words, we say that the ratio is inconclusive.

   NOTE: Remember to take the absolute value of the term a_n before taking its n^{th} root.

   In general, this test is used only if **powers are involved**. Though, it is not necessary for you to use this test every time you see powers. For example, let us take a look at the following contrasting examples.

   (a) Determine whether the series $\sum_{n\to\infty}(-1)^n(\sqrt{n+1}-\sqrt{n})^n$ converges or diverges.

   Note that $a_n = (-1)^n(\sqrt{n+1} - \sqrt{n})^n \geq 0$. Consider

$$\begin{aligned}
\lim_{n\to\infty} |a_n|^{1/n} &= \lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n})^{n^{1/n}} \\
&= \lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) \\
&= \lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n})\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \quad \text{rationalize} \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0
\end{aligned}$$

   Since $\lim_{n\to\infty} |a_n|^{1/n} = 0 < 1$, the given series $\sum(\sqrt{n+1} - \sqrt{n})^n$ converges (in fact absolutely) by Root test.

(b) Determine whether the series $\sum_{n\to\infty}(-1)^n(\sqrt{n+1}+\sqrt{n})^n$ converges or diverges.

We do not NEED Root test for this one, though, we can use if we want. Note that $\lim_{n\to\infty}(\sqrt{n+1}+\sqrt{n})^n == \infty)^\infty = \infty$. This implies that $\lim_{n\to\infty}(-1)^n(\sqrt{n+1}+\sqrt{n})^n$ does not exist. Hence the given series $\sum_{n\to\infty}(-1)^n(\sqrt{n+1}+\sqrt{n})^n$ diverges by divergence test.

If the terms are rational functions of n then this test is very difficult to apply and also fails almost every time. Even for simple example such as $\sum \frac{1}{n}$ this test fails. Indeed if $a_n = \frac{1}{n}$ and we consider

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{1}{n}^{1/n} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = 1$$

using the fact that $\lim_{n\to\infty} n^{1/n} = 1$. This implies that the root test is inconclusive.

Summary of Some Important Tips

1. **Rational terms** are best handled with comparison or limit comparison test with p-series test. NO Root or Ratio test for rational functions of n.
2. **Powers of** n — Root test.
3. **Factorials and the combination of factorials and powers** — Ratio test.
4. Divergence test is inconclusive if $\lim_{n\to\infty} a_n = 0$.
5. CT is inconclusive if (your series) $\leq$ (divergent series) or (you series) $\geq$ (convergent series) or if any of the terms in either sequence are negative.
6. LCT is inconclusive if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ or ∞, or if any of the terms in either sequence are negative.
7. AST is inconclusive if a_n is not decreasing or positive.
8. The Ratio test if inconclusive if $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$.
9. The Ratio test if inconclusive if $\lim_{n\to\infty} |a_n|^{1/n} = 1$.
10. The tests which are only valid for positive terms can be used for the series with negative terms as well by taking the absolute value of the terms. This can help us determining at least the absolute convergence of the series.

Last Remark: Absolute convergence means everything converges, that is, $\sum |a_n|$ converges which further implies that $\sum a_n$ convergence. Thus it can thought of as the "strong convergence". On the other hand, conditional convergence means that the series $\sum a_n$ converges BUT $\sum |a_n|$ diverges. Thus we can think of this convergence as " weak convergence" or "partial convergence".