21-127 Concepts Homework 6: Solutions

5.13 To obtain k 6s from 4 dice there are $\binom{4}{k}$ arrangements in which the 6s could occur. The probability of a 6 is $\frac{1}{6}$ and of a non-six is $\frac{5}{6}$ so $\mathbb{P}(k \text{ 6s}) = \binom{4}{k}(\frac{1}{6})^k(\frac{5}{6})^{n-k}$.

Observe that by the binomial theorem

$$\sum_{k=0}^{4} \binom{4}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{n-k} = \left(\frac{1}{6} + \frac{5}{6}\right)^4 = 1^4 = 1$$

so the probabilities sum to one.

5.21 To choose a rectangle we need to choose its two vertical sides from n possibilities, and then its two horizontal sides from m possibilities, so overall there are $\binom{n}{2}\binom{m}{2}$ possible rectangles.

5.26 Base case $n = 1$: $(x + y)^1 = x + y = \sum_{k=0}^{1} \binom{1}{k} x^k y^{1-k}$

Given for n:

$$\begin{aligned}
(x+y)^{n+1} &= (x+y)(x+y)^n \\
&= (x+y)\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k} && \text{by hypothesis} \\
&= \sum_{k=0}^{n}\binom{n}{k}x^{k+1}y^{n-k} + \sum_{k=0}^{n}\binom{n}{k}x^k y^{n+1-k} \\
&= \sum_{k=1}^{n+1}\binom{n}{k-1}x^k y^{n-k+1} + \sum_{k=0}^{n}\binom{n}{k}x^k y^{n+1-k} && \text{re-indexing first sum} \\
&= x^{n+1} + y^{n+1} + \sum_{k=1}^{n}\left(\binom{n}{k-1} + \binom{n}{k}\right)x^k y^{n+1-k} \\
&= x^{n+1} + y^{n+1} + \sum_{k=1}^{n}\binom{n+1}{k}x^k y^{n+1-k} && \text{by Pascal's formula} \\
&= \sum_{k=0}^{n+1}\binom{n+1}{k}x^k y^{n+1-k}
\end{aligned}$$

5.33 a)

$$\begin{aligned}
6\binom{m}{3} + 6\binom{m}{2} + m &= 6\frac{m!}{3!(m-3)!} + 6\frac{m!}{2!(m-2)!} + m \\
&= 6\frac{m(m-1)(m-2)}{6} + 6\frac{m(m-1)}{2} + m \\
&= m(m-1)(m-2) + 3m(m-1) + m \\
&= m^3 - 3m^2 + 2m + 3m^2 - 3m + m \\
&= m^3
\end{aligned}$$

b)

$$\begin{aligned}
\sum_{i=1}^{n} i^3 &= \sum_{i=1}^{n} 6\binom{i}{3} + 6\binom{i}{2} + i \\
&= 6\sum_{i=1}^{n}\binom{i}{3} + 6\sum_{i=1}^{n}\binom{i}{2} + 6\sum_{i=1}^{n} i \\
&= 6\binom{n+1}{4} + 6\binom{n+1}{3} + \binom{n+1}{2} \qquad \text{summation identity} \\
&= 6\frac{(n+1)n(n-1)(n-2)}{4!} + 6\frac{(n+1)n(n-1)}{3!} + \frac{(n+1)n}{2!} \\
&= \frac{1}{4}n(n+1)[(n-1)(n-2) + 4(n-1) + 2] \\
&= \frac{1}{4}n(n+1)[n^2 + n] \\
&= \frac{1}{4}n^2(n+1)^2
\end{aligned}$$

c) m^3 is the number of ordered lists with repetition of length 3 that can beformed from [m]. $\binom{m}{3}$ is the number of unordered such lists without repetion, each of which can be ordered in 6 ways for a total of $6\binom{m}{3}$. Now we must consider those lists that do have repetition. We could have two instances of one element and one of another; in this case there are $\binom{m}{2}$ ways to choose the elements involved, 2 ways to choose which will occur twice, and then 3 ways to order for a total of $6\binom{m}{2}$. Finally we could have three instances of a single element, for which there are m choices.

10.8 Divide the square into four smaller squares each of side length $\frac{1}{2}$ and use these as our pigeonholes (it doesn't matter to which square we assign the lines dividing it from its neighbours). Now we have five points in four small squares, so there must be some small square containing at least two points. And two points in a small square are most widely separated if placed in opposite corners, in which case they are still within distance $\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2}$ of each other.

Place one point in each corner of the large square and one in the centre.

10.10 Consider the the bottom point of a football field; it must lie within the first 300 yards of our field (to allow at least 100 yards beyond it for the rest of the football field). Divide this 300 yards into three pigeonholes: [0,100),[100,200),[200,300]. Now we are assigning 10 starting points (pigeons) to three pigeonholes, so we must have some pigeonhole containing at least four. So four football fields start in the same interval, for example [100,200) and some must all contain the point at the end of this interval, e.g. 200.

10.24 3^{10} ways; for each person we choose a room.

We use the inclusion-exclusion principle. There are 3^{10} ways to assign 10 people to the three rooms. We must then remove the cases when a room is unoccupied; for a given room this could happen in 2^{10} ways (the number of ways to assign 10 people to the remaining two rooms) and there are 3 rooms so subtract 3.2^{10}. However we have subtracted twice the cases when two rooms stand empty; there are three choices of which these two rooms could be, and for each there is 1 possible arrangement - everyone goes in the one remaining room. So the answer is $3^{10} - 3.2^{10} + 3$.

5.37 This is the number of ways to choose from n people a committee of size k and a subcommittee of size j. The LHS first choses the committee and then the subcommittee from it. The RHS first choses the subcommittee, and then the rest of the committee from the rest of the people.

5.46

$$
\begin{aligned}
\sum_{S\subset[n]}\prod_{i\in S}\frac{1}{i} &= \sum_{S\subset[n]}\prod_{i\in S}\frac{1}{i}\prod_{i\notin S}1 \\
&= \prod_{i=1}^{n}\left(\frac{1}{i}+1\right) \\
&= \prod_{i=1}^{n}\left(\frac{1+i}{i}\right) \\
&= \frac{n+1}{1} \qquad \text{the numerator of each term cancels with the denominator of the next} \\
&= n+1
\end{aligned}
$$

Alternatively you can do a proof by induction.

5.57 We will show $\sum_{k=1}^{n} k.k! = (n+1)!-1$. The RHS counts the number of permutations of $[n+1]$ except for the identity permutation. The summation also counts this set, partitioned by letting $k+1$ be the highest value of i such that element i is not in position i. For such a permutation, elements $k+2,...,n$ are fixed in positions $k+2,...,n$ respectively giving one choice only, and elements $1,...k+1$ go in positions $1,...,k+1$ with the proviso that $k+1$ is not in position $k+1$, which gives us $k.k!$ options. The identity permutation has no element out of place and so is not counted in this sum; all other elements are.

5.64 Proof by strong induction on n, for all k at once.

Base case $n=0$: Unique solution is $m_i = i-1$ for $1 \le i \le k$.

Given below n: Take m_k maximal such that $\binom{m_k}{k} \le n$. Then by induction hypothesis (with $k-1$) we can write $n - \binom{m_k}{k} = \binom{m_1}{1} + ... + \binom{m_{k-1}}{k-1}$. We must check that $m_{k-1} < m_k$; if not then,

$$
\begin{aligned}
n &\le \binom{m_k}{k} + \binom{m_{k-1}}{k-1} \\
&\le \binom{m_k}{k} + \binom{m_k}{k-1} \\
&= \binom{m_k+1}{k} \qquad \text{(Pascal's formula)}
\end{aligned}
$$

which contradicts the maximality of our choice of m_k.

For uniqueness it suffices to show that m_k is uniquely determined, as then by induction hypothesis $n - \binom{m_k}{k}$ has a unique representation that completes the representation of m_k. If we chose a value larger than m_k then by maximality of m_k we would already have a total greater than n. If we chose a value smaller then the greatest sum we could achieve would be,

$$\begin{aligned}\binom{m_k - 1}{k} + \binom{m_k - 2}{k - 1} + ... + \binom{m_k - k}{1} &= \binom{m_k - 1}{m_k - k - 1} + ... + \binom{m_k - k}{m_k - k - 1} \\ &< \binom{m_k - 1}{m_k - k + 1} + ... + \binom{1}{m_k - k + 1} \\ &= \binom{m_k}{m_k - k} \qquad \text{summation identity} \\ &\leq n\end{aligned}$$

so our total is still too small.

10.19 990 keys is sufficent. Give each of 90 people a key to their own room (90 keys total), and the remaining 10 people a key to all 90 rooms (900 keys total). Now given any collection of 90 people, each of the 90 who has a single key goes to his own room, and the remaining rooms can be taken by the remaining people (who each has a key to every room).

990 keys is necessary, because if we have 989 or fewer then as $\frac{989}{90} < 11$ there is some room to which less than 11 people have a key; the number of keys is an integer so at most 10 people have a key to this room. Hence we can pick 90 people omitting these 10, and none of these 90 will have a key to that room, so they cannot each have a room of their own.

10.23 A 4x4 grid is not sufficent to force a monochromatic rectangle; it's not hard to come up with a counterexample colouring.

Given a 5x5 colouring. In the first row there must be a majority of one colour, wlog black; it doesn't matter if we exchange columns so wlog the first three elements of the first row are black - (B,B,B). Now consider the first three elements of the second row. If they are (W,W,W) then in the third row we must have at least two black or two white, giving a rectangle. The second row cannot have two or more black, so the remaining case is one black two white, wlog (W,W,B). Now the third, fourth and fifth rows cannot have either (W,W) or (B,B) in their first two spaces so they must be either (W,B) or (B,W). To avoid a black rectangle involving the first row each must actually be (W,B,W) or (B,W,W). This is two possibilities and there are three remaining rows, so one must occur twice, giving a white rectangle.

10.37 We use inclusion-exclusion. Given a set $S \subset [n]$ of couples, consider the number of arrangements in which they sit next to each other. We can replace each couple in S by a single token (as they must stay together), leaving $2n-|S|$ tokens to arrange in a circle, and there are $(2n-|S|-1)!$ ways to do this. For each couple in S we decide who sits on the left, giving $2^{|S|}$ additional choices. Now by inclusion-exclusion the number of arrangements where no couple sits together is,

$$\sum_{S\subset[n]}(-1)^{|S|}(2n-|S|-1)!2^{|S|} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}(2n-k-1)!2^k$$