21-127 Concepts Homework 5: Solutions

4.20 a) First check injectivity:

$$\begin{aligned}
f((x,y)) = f((u,v)) &\Rightarrow (ax-by, bx+ay) = (au-bv, bu+av) \\
&\Rightarrow ax-by = au-bv \wedge bx+ay = bu+av \\
&\Rightarrow a(x-u) = b(y-v) \wedge b(x-u) = a(v-y) \\
&\Rightarrow ab(x-u) = b^2(y-v) \wedge ab(x-u) = -a^2(y-v) \\
&\Rightarrow (b^2+a^2)(y-v) = 0 \\
&\Rightarrow y-v = 0 \qquad (a^2+b^2 \neq 0) \\
&\Rightarrow y = v \\
&\Rightarrow a(x-u) = 0 \wedge b(x-u) = 0 \\
&\Rightarrow x-u = 0 \qquad (\neg a = b = 0) \\
&\Rightarrow x = u \\
&\Rightarrow (x,y) = (u,v)
\end{aligned}$$

Now check surjectivity; given $(r,s) \in \mathbb{R}^2$ we seek (x,y) such that $f((x,y)) = (r,s)$. Solving the resulting simultaneous equation suggests $(x,y) = \left(\frac{ar+bs}{a^2+b^2}, \frac{as-br}{a^2+b^2}\right)$ so we check this works:

$$\begin{aligned}
f\left(\left(\frac{ar+bs}{a^2+b^2}, \frac{as-br}{a^2+b^2}\right)\right) &= \left(a\frac{ar+bs}{a^2+b^2} - b\frac{as-br}{a^2+b^2}, b\frac{ar+bs}{a^2+b^2} + a\frac{as-br}{a^2+b^2}\right) \\
&= \left(\frac{a^2r+abs-abs+b^2r}{a^2+b^2}, \frac{abr+b^2s+a^2s-abr}{a^2+b^2}\right) \\
&= (r,s)
\end{aligned}$$

b) Our work in (a) suggests inverse $g(x,y) = \left(\frac{ax+by}{a^2+b^2}, \frac{ay-bx}{a^2+b^2}\right)$, we must check that $f \circ g = id$ and $g \circ f = id$,

$$\begin{aligned}
f(g((x,y))) &= f(\left(\frac{ax+by}{a^2+b^2}, \frac{ay-bx}{a^2+b^2}\right)) \\
&= \left(a\frac{ax+by}{a^2+b^2} - b\frac{ay-bx}{a^2+b^2}, b\frac{ax+by}{a^2+b^2} + a\frac{ay-bx}{a^2+b^2}\right) \\
&= \left(\frac{a^2x+aby-aby-b^2x}{a^2+b^2}, \frac{abx+b^2y+a^2y-abx}{a^2+b^2}\right) \\
&= (x,y)
\end{aligned}$$

Note that this calculation is almost identical to the one we used to check surjectivity.

$$\begin{aligned} g(f((x,y))) &= g((ax-by, bx+ay)) \\ &= \left(\frac{a(ax-by)+b(bx+ay)}{a^2+b^2}, \frac{a(bx+ay)-b(ax-by)}{a^2+b^2}\right) \\ &= \left(\frac{a^2x+b^2x}{a^2+b^2}, \frac{a^2y+b^2y}{a^2+b^2}\right) \\ &= (x,y) \end{aligned}$$

c) f describes a rotation about the origin. For example for $a = 0$, $b = 1$ we have $f((x,y)) = (-y, x)$ which is a rotation by 90 degrees anticlockwise.

4.21 Define f from {even size subsets} to {odd size subsets} by $f(A) = A \triangle \{1\}$. So if 1 was absent we add it, and if it was present we remove it; either way we change the parity of $|A|$ from even to odd. We can also define g from {odd size subsets} to {even size subsets} by $g(A) = A \triangle \{1\}$. Then

$$f(g(A)) = f(A \triangle 1) = (A \triangle 1) \triangle 1 = A \triangle (1 \triangle 1) = A \triangle \emptyset = A$$

and

$$g(f(A)) = f(A \triangle 1) = (A \triangle 1) \triangle 1 = A \triangle (1 \triangle 1) = A \triangle \emptyset = A$$

So g is the inverse of f and hence f is bijective.

Note that even though f and g are defined in the same way they are *not* the same function because they have different domains and codomains.

4.24 No, for example $f(x) = g(x) = x$. Then $fg(x) = f(x)g(x) = x^2$ and this is not surjective; for example -1 is not in its image.

4.39 Observe that $f(f(x)) = f(a(x+b)-b) = a([a(x+b)-b]+b)-b = a^2(x+b)-b$. We guess that $f^n(x) = a^n(x+b)-b$ and try to prove this by induction.

Base case $n = 1$: immediate.

Given for n,

$$\begin{aligned} f^{n+1}(x) &= f(f^n(x)) \\ &= f(a^n(x+b)-b) \\ &= a([a^n(x+b)-b]+b)-b \\ &= a^{n+1}(x+b)-b \end{aligned}$$

5.3 Observe that the numbers on opposite sides of a die sum to 7 (there are either 1 & 6, 2 & 5 or 3 & 4). So for any roll totalling x we can turn over the two dice and get a roll totalling $(7+7)-x=14-x$. The act of turning over is a bijection (it is its own inverse) so there are equal numbers of rolls totalling x and totalling $14-x$.

5.6 Number the elements of A and B from 1 to n and regards bijection from A to B as bijections from $[n]$ to $[n]$. Such a bijection is just a permutation, and the number of these is $n!$.

5.17 Note $n, m < k$ and without loss of generality $n \leq m$. Dividing through by $n!$ we get $1+\frac{m!}{n!}=\frac{k!}{n!}$. But $\frac{m!}{n!}$ divides $\frac{k!}{n!}$ and the only way this is possible is if $\frac{m!}{n!}=1$. Then $m!=n!$ so $m=n$; and $k(k-1)..(n+1)=\frac{k!}{n!}=2$ so $k=2$ and $n=1$. This gives $(n,m,k)=(1,1,2)$ as the only possibility.

4.15 Proof by induction on k,

Base case $k=1$: We can weigh an object of weight 1 with a weight of 1.

Given for k: We can weight $1,...,\frac{3^k-1}{2}$ already by hypothesis. We also now have a new weight 3^k. By placing this on one side and the previous weight combinations on the other we can get weights 3^k-1 through $3^k-\frac{3^k-1}{2}=\frac{3^k+1}{2}$. So this gives us all weights up to 3^k-1, and we have 3^k just taking the last weight by itself. Finally we have weights 3^k+1 through $3^k+\frac{3^k-1}{2}=\frac{3^{k+1}-1}{2}$ by putting the 3^k weight on the same side as our earlier combinations.

4.34 a) True. Given that h is injective,

$$f(x)=f(y) \Rightarrow g(f(x))=g(f(y)) \Rightarrow h(x)=h(y) \Rightarrow x=y$$

so f is injective.

b) False. For example $A=\{1\}$, $B=\{2,3\}$, $C=\{4\}$, $f(1)=2$, $g(2)=g(3)=4$. Then g is not injective but $h(1)=4$ so h is.

c) False. Using the same counterexample as (b).

d) True. Given $z \in C$, h is surjective so we have $x \in A$ such that $g(f(x)) = h(x) = z$. Define $y = f(x)$; then $y \in B$ and $g(y) = z$.

4.50 We want to find all the A_n for n odd, as these are the places we will use g^{-1}; everywhere else we will use f. We observe $B_0 = B \backslash im(f) = \{0\}$. Then A_1 is all the members of A with exactly one ancestor, i.e. the children of members of B_0 (which have no ancestors), so $A_1 = \{g(0)\} = \{\frac{1}{2}\}$. Then A_3 contains the grandchildren of these children so $A_3 = \{g(f(\frac{1}{2}))\} = \{g(\frac{1}{2})\} = \{\frac{3}{4}\}$. Continuing in the vein $A_{2n-1} = \{1 - \frac{1}{2^n}\}$.So,

$$h(x) = \begin{cases} g^{-1}(x) = 2x - 1 & x = 1 - \frac{1}{2^n} \text{ for some } n \in \mathbb{N} \\ f(x) = x & \text{otherwise} \end{cases}$$

4.51 We could take an injection in each direction and then use the Schroeder-Bernstein method as in question 4.50 to get a bijection. But it's faster just to make up a function that fills in the gaps:

$$h(x) = \begin{cases} 0 & x = \frac{1}{2} \\ \frac{1}{n-2} & x = \frac{1}{n} \text{ for } n \in \mathbb{N}, n \geq 3 \\ x & \text{otherwise} \end{cases}$$

Extra 1 In any disc take two distinct points (x, y) and (u, v). Now by Theorem 1.10 we can find a rational between any two real numbers; take rationals p between x and u; and q between y and v. Then (p, q) lies between (x, y) and (u, v) and hence inside the disc. This tells us that inside any disc there is a member of $\mathbb{Q}^2$.

Given a disjoint arrangement of discs in $\mathbb{R}^2$, choose a member of $\mathbb{Q}^2$ inside each; this defines a function from {discs} to $\mathbb{Q}^2$ and the function is injective because the discs are disjoint. Hence $|\{\text{discs}\}| \leq |\mathbb{Q}^2| = |\mathbb{N}|$ so the collection of discs is countable.

Extra 2 a) We saw in class that $|\{\text{functions } \mathbb{N} \to \mathbb{N}\}| = |\mathbb{P}(\mathbb{N})|$, so certainly $|\{\text{non-decreasing functions} \mathbb{N} \to \mathbb{N}\}| \leq |\mathbb{P}(\mathbb{N})|$. Given $A \in \mathbb{P}(\mathbb{N})$ define a function $f : \mathbb{N} \to \mathbb{N}$ iteratively by $f(1) = 1$ and,

$$f(n+1) = \begin{cases} f(n) & n \notin A \\ f(n) + 1 & n \in A \end{cases}$$

Note that the resulting function is non-decreasing. Also this process of changing a set into a function constitutes an injective function from $\mathbb{P}(\mathbb{N})$

to {non-decreasing functions $\mathbb{N} \to \mathbb{N}$} because we can find whether n was in A by checking whether $f(n+1) - f(n) = 1$. Hence we have $|\mathbb{P}(\mathbb{N})| \leq |\{\text{non-decreasing functions } \mathbb{N} \to \mathbb{N}\}|$ so by Schroeder-Bernstein $|\{\text{non-decreasing functions } \mathbb{N} \to \mathbb{N}\}| = |\mathbb{P}(\mathbb{N})|$.

b) There are certainly at least $|\mathbb{N}|$-many non-increasing functions, for example the constant functions. Observe that a non-increasing function can only decrease a finite number of times, and must eventually stabilise, say by n so $\forall m \geq n : f(m) = f(n)$. Now we can regard this function as a finite list $\{f(1), f(2), ..., f(n)\}$, giving us a mapping from {non-increasing functions $\mathbb{N} \to \mathbb{N}$} to {finite lists of naturals}; this mapping is injective because the function is recoverable from the list (just take $f(m) = f(n)$ for $m > n$). Hence $|\{\text{non-increasing functions } \mathbb{N} \to \mathbb{N}\}| \leq |\{\text{finite lists of naturals}\}| = |\mathbb{N}|$ (saw in class). So by Schroeder Bernstein $|\{\text{non-increasing functions } \mathbb{N} \to \mathbb{N}\}| = |\mathbb{N}|$.

Extra 3 We have injective $f : \mathbb{R} \to \mathbb{R}^2$ given by $f(x) = (x, 0)$. And we have injective $f : \mathbb{R}^2 \to \mathbb{R}$ given by taking $(c_n...c_1.a_1a_2a_3...., d_n...d_1.b_1b_2....)$ to $c_nd_nc_{n-1}...d_2c_1d_1.a_1b_1a_2b_2....$ I.e. we interleave their decimal expansions, and this will retain all the information from the original two reals so it is injective. Whence by Schroder Bernstein $|\mathbb{R}^2| = |\mathbb{R}|$.

Upon first seeing this proof, the great mathematician Georg Cantor wrote *'Je le vois, mais je ne crois pas'* - 'I see it but I don't believe it'.