21-127 Concepts Homework 4: Solutions

4.5 Any set with at least two elements. If the set is empty the only bijection is the vacuous one of doing nothing, which is the identity (it satisfies $\forall x f(x) = x$). If there is a single element, say a then any bijection must take a to a and hence be the identity. If there are at least two distinct elements, say a and b then we can define a non-identity bijection by,

$$f(x) = \begin{matrix} b & x = a \\ a & x = b \\ x & x \neq a, b \end{matrix}$$

4.11 In both cases multiplication by 2 is injective as $2x = 2y \Rightarrow x = y$. From $\mathbb{R}$ to $\mathbb{R}$ it is surjective because for any $x \in \mathbb{R}$ we have $2.\frac{x}{2} = x$ so x is in the image. But from $\mathbb{Z}$ to $\mathbb{Z}$ it is not because 3 is not in the image (there is no integer that when multiplied by 2 gives 3).

4.23 a) This is not injective as $f(0) = f(1) = 1$. It is surjective as is suggested by its graph; as $x \to \infty$, $f(x) \to \infty$ and as $x \to -\infty$, $f(x) \to -\infty$. It is a bijection from $(1, \infty)$ to $(1, \infty)$.

b) This is not injective as $f(0) = f(4) = 1$. It is not surjective as the image of cos is restricted to $[-1, 1]$ and so does not contain, for example, 2. It is a bijection from $(0, 1)$ to $(0, 1)$.

4.29 a) $f(2) = f(\frac{1}{2}) = \frac{2}{5}$ so f is not injective. $g(1) = g(-1) = \frac{1}{2}$ so g is not injective. We prove that h is injective:

$$\begin{aligned} h(x) = h(y) &\Rightarrow \frac{x^3}{1+x^2} = \frac{y^3}{1+y^2} \\ &\Rightarrow x^3(1+y^2) = y^3(1+x^2) \\ &\Rightarrow (x^3 - y^3) + (x^3y^2 - y^3x^2) = 0 \\ &\Rightarrow (x-y)(x^2+xy+y^2) + (x-y)x^2y^2 = 0 \\ &\Rightarrow x = y \vee x^2 + xy + y^2 + x^2y^2 = 0 \\ &\Rightarrow x = y \vee (x + \frac{y}{2})^2 + \frac{3}{4}y^2 + x^2y^2 = 0 \end{aligned}$$

Notice that every part of this sum is a square and so ≥ 0. So the only way the sum can be 0 is if every part is 0, i.e. $x = y = 0$. Hence either way $x = y$.

b) Note that for all x,

$$\begin{aligned}
&(x - \frac{1}{2})^2 \geq 0 \\
\Rightarrow &x^2 - x + \frac{1}{4} \geq 0 \\
\Rightarrow &x^2 + 1 \geq x \\
\Rightarrow &1 \geq \frac{x}{1 + x^2} \\
\Rightarrow &f(x) \leq 1
\end{aligned}$$

So 2 is not in the image of f, so f is not surjective.

Note that the numberator and denominator of g are always ≥ 0 so $g(x)$ is always positive; whence g does not have -1 in its image and is not surjective.

4.37 Given that $f \circ f$ is injective,

$$f(x) = f(y) \Rightarrow f(f(x)) = f(f(y)) \Rightarrow x = y$$

So f is injective.

4.47 We have a bijection from $\mathbb{N}$ to the even naturals given by $f(x) = 2x$. And we have a bijection to the odd naturals $g(x) = 2x - 1$.