21-127 Concepts Homework 3: Solutions

3.23 There is a problem in the first inductive step, going from $n = 0$ to $n = 1$. We use that $a^{n-1} = 1$ where here $n = 0$ so $n - 1 = -1$ and we have not proved that $a^{-1} = 1$ (it is in fact false for most a).

3.27 By induction on n:

Base case $n = 1$: LHS $= \frac{1}{1.4} = \frac{1}{4}$

RHS $= \frac{1}{4}$

Inductive step: given for n

$$\begin{aligned}
\sum_{i=1}^{n+1} \frac{1}{(3i-2)(3i+1)} &= \sum_{i=1}^{n} \frac{1}{(3i-2)(3i+1)} + \frac{1}{(3n+1)(3n+4)} \\
&= \frac{n}{3n+1} + \frac{1}{(3n+1)(3n+4)} \\
&= \frac{n(3n+4)+1}{(3n+1)(3n+4)} \\
&= \frac{3n^2+4n+1}{(3n+1)(3n+4)} \\
&= \frac{(3n+1)(n+1)}{(3n+1)(3n+4)} \\
&= \frac{n+1}{3n+4}
\end{aligned}$$

3.44 It is possible for $n \in \{3, 6, 9, 10, 12, 13, 15, 16, 18\}$ and not for $n \in \{1, 2, 4, 5, 7, 8, 11, 14, 17\}$ by inspection. We will prove by strong induction that it is possible for all $n \geq 18$

If $n = 18$ take $3 + 3 + 3 + 3 + 3 + 3$

If $n = 19$ take $10 + 3 + 3 + 3$

If $n = 20$ take $10 + 10$

If $n \geq 21$ then $n - 3 \geq 18$ so by strong induction hypothesis we know that $n-3$ is possible. Take this combination and add an additional 3 to get n.

3.49 a) We first compute some small values:

n	3^n	2^{n+1}	holds?
1	3	4	no
2	9	8	yes
3	27	16	yes
4	81	32	yes

We now prove by induction that the inequality holds for all $n \geq 2$
Base case: done above
Given for n, $3^{n+1} = 3.3^n \geq 3.2^{n+1} \geq 2.2^{n+1} = 2^{n+2}$

b)

n	2^n	$n+1^2$	holds?
1	2	4	no
2	4	9	no
3	8	16	no
4	16	25	no
5	32	36	no
6	64	49	yes
7	128	64	yes

We prove by induction that the inequality holds for $n \geq 6$
Base case: done above
Given for n

$$\begin{aligned} 2^{n+1} &= 2.2^n \\ &\geq 2(n+1)^2 \\ &= 2n^2 + 4n + 2 \\ &\geq n^2 + 4n + 4 \\ &= (n+2)^2 \end{aligned}$$

c)

n	3^{n+1}	n^4	holds?
1	9	1	yes
2	27	16	yes
3	81	81	yes
4	243	256	no
5	729	625	yes
6	2187	1296	yes

We prove by induction that the inequality holds for $n \geq 5$
Base case: done above
Given for n

$$\begin{aligned}
3^{n+2} &= 3.3^{n+1} \\
&\geq 3n^4 \\
&= n^4 + \frac{4}{5}n^4 + \frac{6}{25}n^4 + \frac{4}{125}n^4 + \frac{1}{625}n^4 + \frac{579}{625}n^4 \\
&\geq n^4 + 4n^3 + 6n^2 + 4n^3 + 1 \qquad (n \geq 5) \\
&= (n+1)^4
\end{aligned}$$

Note that we had to use that $n \geq 5$ in our induction step, as it is not true that the inequality holding for $n = 3$ implies that it holds for $n = 4$.

d)

n	$n^3 + (n+1)^3$	$(n+2)^3$	holds?
1	9	27	no
2	35	64	no
3	91	125	no
4	189	216	no
5	341	343	no
6	559	512	yes
7	855	729	yes

We prove by induction that the inequality holds for $n \geq 6$
Base case: done above

Given for n

$$\begin{aligned}(n+1)^3+(n+2)^3 &= (n^3+3n^2+3n+1)+(n^3+6n^2+12n+8)\\ &= 2n^3+9n^2+15n+6\\ &= n^3+9n^2+(15n+\frac{12}{36}n^3)+(6+\frac{21}{216}n^3)+\frac{41}{72}n^3\\ &\geq n^3+9n^2+27n+27 \qquad (n\geq 6)\\ &= (n+3)^2\end{aligned}$$

3.58 a) Covered in lectures

b) Proof by induction on k

Base case $k=1$: a 2x2 chessboard missing one square is just a single L-tile.

Given for k: Given a 2^{k+1} by 2^{k+1} chessboard with one square missing, regard it as four 2^k by 2^k quadrants. One of these has the missing square; place an L-tile in the centre of the 2^{k+1} by 2^{k+1} board covering exactly one square from each of the other three quadrants. Now we have four 2^k by 2^k quadrants each missing a single square, and by hypothesis we can tile each of these, yielding an overall tiling.

4.10 For any $y\in\mathbb{R}$, $ax+b=y$ has a unique solution, namely $x=\frac{y-b}{a}$. That there is at least one gives surjectivity; that there is at most one gives injectivity. Likewise for g.

$$\begin{aligned}h(x) &= g(f(x))-f(g(x))\\ &= g(ax+b)-f(cx+d)\\ &= c(ax+b)+d-a(cx+d)-b\\ &= bc+cd-ad-ab\end{aligned}$$

So the function is not surjective as nothing maps to points in $\mathbb{R}$ other than $bc+cd-ad-ab$ and it is not injective as more than one point (in fact all of them) maps to $bc+cd-ad-ab$.

4.12 a) False, e.g. $f(x)=-arctan(x)$

b) False, e.g. $f(x)=0$

c) False, e.g.

$$f(x)=\begin{matrix}1/x & x\neq 0\\ 0 & x=0\end{matrix}$$

d) True. Every real number appears in the image, so there is no bound on the absolute value of numbers in the image.

e) False, e.g. $f(x) = |x|$

3.31 We will show $\prod_{i=2}^{n}(1 - \frac{1}{i^2}) = \frac{n+1}{2n}$

Base case $n = 2$: LHS $= 1 - \frac{1}{4} = \frac{3}{4}$

RHS $= \frac{3}{4}$

Given for n,

$$\begin{aligned}
\prod_{i=2}^{n+1}\left(1 - \frac{1}{i^2}\right) &= \prod_{i=2}^{n}\left(1 - \frac{1}{i^2}\right) \cdot \left(1 - \frac{1}{(n+1)^2}\right) \\
&= \left(\frac{n+1}{2n}\right)\left(\frac{(n+1)^2 - 1}{(n+1)^2}\right) \\
&= \frac{(n+1)(n^2 + 2n)}{2n(n+1)^2} \\
&= \frac{(n+2)}{2(n+1)}
\end{aligned}$$

3.38 We will show by induction on n that player 2 can win a game in which the target is $4n$.

Base case $n = 0$: player 2 has won before the game even starts.

Given for n: By hypothesis player 2 can get to $4n$. Then player 1 must say 1,2 or 3. In response player 2 should say 3,2 or 1 respectively giving a total of $4n + 4$.

3.39 Observe that in going from a_n to $a_(n + 1)$ we add an outer ring of $6n$ dots. And we start with a_1=1 So

$$\begin{aligned}
a_n &= 1 + \sum_{i=1}^{n-1} 6i \\
&= 1 + 6.\frac{1}{2}n(n-1) \\
&= 3n^2 - 3n + 1
\end{aligned}$$

And

$$\begin{aligned}\sum_{k=1}^{n} a_k &= \sum_{k=1}^{n} 3n^2 - 3n + 1 \\ &= 3.\frac{1}{6}n(n+1)(2n+1) - 3.\frac{1}{2}n(n+1) + n \\ &= \frac{1}{2}(2n^3 + 3n^2 + n - 3n^2 - 3n + 3n) \\ &= n^3\end{aligned}$$

3.59 Note that an L-tile has an area of 3, so to be coverable a rectangle must have an area divisible by 3; so we will only consider rectangles with dimensions $(3m, n)$

Note that we can build a (3,2) rectangle by combining two L-tiles. $(3m, 2n)$: Possible, splitting into multiple (3,2) rectangles.

$(3, 2n + 1)$: Not possible, proof by induction on n. For $n = 0$ $(3, 1)$ is clearly impossible. Given that $(3, 2n + 1)$ is impossible, the only way to start covering one end of $(3, 2n+3)$ is using a pair of L-tiles forming a $(3, 2)$ rectangle (check the cases) and then we are left with a $(3, 2n + 1)$ rectangle which we know by hypothesis is impossible to cover.

$(6, 2n + 1)$: Possible for $n \geq 1$, proof by induction on n. For $n = 1$ we have seen that $(6, 3)$ is possible. Given a covering for $(6, 2n + 1)$ add two $(3, 2)$ rectangles to extend to a covering of $(6, 2n + 3)$

$(9, 2n+1)$: Possible for $n \geq 2$ only. We have seen that $(9, 3)$ is not possible. Observe that $(9, 5)$ is possible by an explicit construction (it's not built up from smaller blocks, but it's not hard to find one). Then by induction we can add $(3, 2)$ rectangles to cover any $(9, 2n + 1)$ for larger n.

$(6m, 2n + 1)$: Possible for $n \geq 1$ using multiple copies of the $(6, 2n + 1$ covering.

$(6m + 3, 2n + 1)$: Possible for $n \geq 2$ only. We have seen that $(6m + 3, 3)$ is not possible. Otherwise split into one copy of $(9, 2n + 1)$ and some copies of $(6, 2n + 1)$ and use the coverings for these.

So in summary a covering of (p, q) is possible if and only if pq is divisible by 3, except if $p = 3$ and q is odd or vice-versa, in which case it is not possible.

3.65 We prove by induction on n that if there are n thieves then they will be revealed on the nth day.

Base case $n = 1$: If there is only a single thief, his master will see that there are no other thieves and, knowing that there is at least one, conclude that his apprentice is a thief and denounce him on day 1.

Given for n. If there are $n+1$ thieves then each master whose apprentice is a thief will see n other thieves. He knows by the inductive hypothesis that if there were n thieves then they would be revealed on the nth day. At the end of the nth day he sees that this hasn't happened so he concludes that there must be $n+1$ thieves including his own apprentice, and denounces him on the $n+1$th day.

Extra 1 Say the large disc has radius R, and the first smaller disc has radius $r < R$. The small disc is only able to cover points a distance $2r$ apart, so it cannot cover opposite points on the edge of the large disc; hence it can cover less than half the circumference of the large disc. Likewise the second small disc covers less than half the circumference of the large one, so between them they cannot cover the whole circumference, and so cannot cover the whole disc.

Extra 2 Proof by induction on n.

Base case $n = 0$: $e^x \geq 1$ because $x \geq 0$.

Given for n,

$$\begin{aligned}
& e^x \geq 1 + x + \frac{x^2}{2} + ... + \frac{x^n}{n!} \\
\Rightarrow & \int_0^x e^t dx \geq \int_0^x 1 + t + \frac{t^2}{2} + ... + \frac{t^n}{n!} dx \\
\Rightarrow & e^x - 1 \geq x + \frac{x^2}{2} + ... + \frac{x^{n+1}}{(n+1)!} \\
\Rightarrow & e^x \geq 1 + x + \frac{x^2}{2} + ... + \frac{x^{n+1}}{(n+1)!}
\end{aligned}$$