21-127 Concepts Homework 2: Solutions

1.4 For a given perimeter p a rectangle of length x has area $x(\frac{p}{2} - x)$ and a square has area $(\frac{p}{4})^2$. Now,

$$\begin{aligned} x \neq \frac{p}{4} &\Rightarrow (x - \frac{p}{4})^2 > 0 \\ &\Rightarrow x^2 - 2\frac{p}{4} + (\frac{p}{4})^2 > 0 \\ &\Rightarrow (\frac{p}{4})^2 > x(\frac{p}{2} - x) \end{aligned}$$

So taking any side length other than $\frac{p}{4}$ (a square) yields an area smaller than that of the square.

2.4 a) There is $x \in A$ such that for all $b \in B$, $x \geq b$

b) For all $x \in A$ there is some $b \in B$ satisfying $b \leq x$

c) There are distinct $x, y \in \mathbb{R}$ such that $f(x) = f(y)$

d) There is some $b \in \mathbb{R}$ that is not in the image of f

e) There are some $x, y \in \mathbb{R}$ and some $\epsilon \in P$ such that for every $\delta \in P$ we have $|x - y| < \delta$ but $|f(x) - f(y)| \geq \epsilon$

f) There is $\epsilon \in P$ such that for all $\delta \in P$ there are $x, y \in \mathbb{R}$ such that $|x - y| < \delta$ but $|f(x) - f(y)| \geq \epsilon$

2.21 Negation: there is some integer $n > 0$ such that for every real number $x > 0$ we have $x \geq 1/n$. The original statement is true; its negation is false.

2.30 a) We need to turn over all the vowels (obviously) and all the even numbers, because if these had a vowel on the other side they would contradict the statement. We do not need to turn over any others; in particular there is no need to check the odd numbers, as the truth of the statement is not affected by either vowel-odd or consonant-odd.

b) We must turn over every card, as either vowel-even or consonant-odd would contradict the statement.

2.31 a) Believable. If I have no five-legged dogs the statement is vacuously true.

b) Believable. This is satisfied by my having no five-legged dogs, which is highly plausible!

c) Unbelievable. This claims I have at least one five-legged dog.

d) Unbelievable. As (c).

2.48 a) False, e.g. taking $x = 1$ is a counter-example as $P(1)$ holds but not $Q(1)$.

b) True. $(\forall x \in \mathbb{Z})(P(x))$ is false (not every integer is odd) so the whole implication is automatically true, regardless of the truth of the right-hand side.

2.51 a) Covered in lectures.

b) First we show $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$. Given $x \in A \cap (B \cup C)$ then have both $x \in A$ and $x \in B \cup C$. So x is in A and also in either B or C. So x is in either $A \cap B$ or $A \cap C$. So $x \in (A \cap B) \cup (A \cap C)$.

Now we show $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$. Given $x \in (A \cap B) \cup (A \cap C)$ then either $x \in (A \cap B)$ or $x \in (A \cap C)$. In either case we have $x \in A$. Also in the first case we have $x \in B$ and in the second $x \in C$ so either way $x \in B \cup C$. Combining these two facts, we get $x \in A \cap (B \cup C)$.

1.22 Let (a, b) denote the amount of wine and water in a glass. Initially we have $(x, 0)$ in glass 1 and $(0, x)$ in glass 2 . After the first step we have $(x-1, 0)$ in glass 1 and $(1, x)$ in glass 2. We then move $(\frac{1}{x+1}, \frac{x}{x+1})$ from glass 2 to glass 1, resulting in $(\frac{x^2}{x+1}, \frac{x}{x+1})$ in glass 1 and $(\frac{x}{x+1}, \frac{x^2}{x+1})$. We observe there is $\frac{x}{x+1}$ water in glass 1 and wine in glass 2.

Alternatively note that both glasses start and end containing x ounces of liquid. So if we end with y ounces of water in glass 1, there must be $x - y$ left in glass 2, leaving room for $x - (x - y) = y$ ounces of wine in glass 2.

1.25 The possible products of three integers yielding 36 are (1,1,36), (1,2,18), (1,3,12), (1,4,9), (1,6,6), (2,2,9), (2,3,6), (3,3,4). The sums of each of these are 38, 21, 16, 14, 13, 13, 11, 10. The sum does not uniquely determine the ages so they must be either (1,6,6) or (2,2,9), both of which have sum 13. Finally, there is a single 'eldest daughter', which rules out (1,6,6) leaving (2,2,9).

1.35 We will show that it is $\{(x, y) \in \mathbb{R}^2 : xy > 0\}$.

Firstly, if $xy > 0$ then,

$$\begin{aligned} (x-y)^2 \geq 0 &\Rightarrow x^2 - 2xy + y^2 \geq 0 \\ &\Rightarrow x^2 + y^2 \geq 2xy \\ &\Rightarrow \frac{x}{y} + \frac{y}{x} \geq 2 \end{aligned}$$

as dividing by xy positive preserves the inequality.

On the other hand if $xy < 0$ then exactly one of x and y is negative so $\frac{x}{y} < 0$ and $\frac{y}{x} < 0$. Hence $\frac{x}{y} + \frac{y}{x} < 0$ so the inequality does not hold.

1.50 a) Given $y \in f(C \cap D)$ then we must have $y = f(x)$ for some $x \in C \cap D$. Now $x \in C$ so $y = f(x) \in f(C)$; likewise $x \in D$ so $y = f(x) \in f(D)$. Combining these we have that $y \in f(C) \cap f(D)$.

b) There are many examples, we just need f appropriately non-injective, e.g. $C = \{0\}$, $D = \{1\}$, $f(x) = 0$ for all x. Then $C \cap D = \emptyset$ so $f(C \cap D) = \emptyset$ but $f(C) = \{0\}$ and $f(D) = \{0\}$ so $f(C \cap D) = \{0\}$.

2.16 a) We need $f(x) = g(x) + h(x)$, whence also $f(-x) = g(-x) + h(-x) = g(x) - h(x)$. Solving for g and h we get $g(x) = \frac{f(x)+f(-x)}{2}$ and $h(x) = \frac{f(x)-f(-x)}{2}$ to be necessary. Now we must check these are sufficient; i.e. that they satisfy all the requirements.

$$g(x) + h(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = f(x)$$
$$g(-x) = \frac{f(-x) + f(x)}{2} = g(x)$$
$$h(-x) = \frac{f(-x) - f(x)}{2} = -\frac{f(x) - f(-x)}{2} = -h(x)$$

b) Even powers satisfy $(-x)^{2n} = x^{2n}$ and odd powers satisfy $(-x)^{2n+1} = -x^{2n+1}$ so g will be the sum of the even powers and h will be the sum of the odd powers.

2.39 First we show that the conditions are necessary. Each day the particle moves one step left, right, up or down; so either a or b rises or falls by one; so $|a| + |b|$ either rise or falls by one. So in k days it can rise to at most k, requiring $|a| + |b| \leq k$. Also each day the parity of $|a| + |b|$ changes, as does that of k (the day number); they start equal at $(a, b) = (0, 0)$ and $k = 0$ so they must finish equal too.

Now we show the conditions are sufficient; i.e. that if the conditions are satisfied there is some path the particle can follow to reach (a, b) by day k. Let us first travel a steps horizontally (left if a is negative, right if positive) to $(a, 0)$ by day $|a|$; then we go b steps vertically to (a, b) by day $|a| + |b|$. We know we have time for this as $|a| + |b| \leq k$. Now we have $k - |a| - |b|$ days remaining, and by equal parity this number is even. So we can spend any remaining time osciallating between (a, b) and $(a, b+1)$ and end on (a, b) for day k.

2.40 a) Each domino covers one white and one black square. There are 30 white and 32 black squares left on the chess board so we cannot cover them all with dominoes (there will be two black squares left over).

b) There are 60 squares left and a T-shape covers 4, so would have to use 15 T-shapes. Each T-shape has an odd number of white squares (1 or 3) so we would cover an odd number of white squares. But there are 30 white squares left on the chessboard, contradiction.

Extra The trick to avoiding massive complexity is to focus on what we know rather than on what we don't know. If the colours alternate between red (R) and blue (B) then the colouring looks like ...RBRBRBR... and we have many arithmetic progressions (APs) of both colours. Otherwise we can find adjacent numbers with the same colour; wlog they are both red: .....RR.... . To avoid a red AP the numbers to each side must be blue: ....BRRB... . To avoid blue APs of difference three we must have .R..BRRB..R. To avoid red APs of difference two we need .R.BBRRBB.R. To avoid blue APs we must have .RRBBRRBBRR. . But this contains a red AP: .RBBRRBBRR. .