A well-ordered set is a structure of the form $(S, \leq)$ such that
- $\leq$ is a partial order on $S$ (see Assignment 4),
- Every nonempty subset of $S$ has a $\leq$-smallest element.

If $(S, \leq)$ is a well-ordered set, we may express this by saying that the relation $\leq$ well-orders $S$.

(a) Prove that $(S, \leq)$ is a well-ordered set, then $\leq$ is a linear order.

Solution. If $(S, \leq)$ is a well-ordered set and $a, b \in S$, then the set $\{a, b\}$ has a $\leq$-smallest element, so $a \leq b$ or $b \leq a$.

(b) Suppose that $(S, \leq)$ is a well-ordered set and $P$ is a property such that
- (i) The smallest element of $S$ satisfies $P$,
- (ii) If $a$ is an element of $S$ and every element of the set
  \[ \{ x \in S \mid x < a \} \]
  satisfies $P$, then the element $A$ satisfies $P$ as well.

Prove that every element of $S$ satisfies $P$.

[Note: This is called the Induction Principle, and it is what makes well-ordered sets important.]

Solution. Suppose that not every element of $S$ satisfies $P$, and let $a$ be the $\leq$-smallest element of $S$ that does not satisfy $P$. Then every element of $\{ x \in S \mid x < a \}$ satisfies $P$, so by (ii), $a$ satisfies $P$. This contradicts the choice of $a$.

(c) Prove that a structure is well-ordered if and only if it does not contain infinite descending chains; that is, prove that a linearly ordered set $(S, \leq)$ is a well-ordered set if and only if there does not exist a sequence $a_0, a_1, a_2, \ldots$ of elements of $S$ such that $a_0 > a_1 > a_2 > \ldots$.

Solution. $(\Rightarrow)$: If $a_0 > a_1 > a_2, \ldots$ is an infinite descending sequence in $S$, then the set $\{ a_0, a_1, a_2, \ldots \}$ does not have a minimum element, so $S$ is not well-ordered.

$(\Leftarrow)$: Suppose that $S$ is not well-ordered, and fix a nonempty $A \subseteq S$ that does not have a minimum element. Fix $a_0 \in A$. Since $a_0$ is not a minimum element of $A$, there exists $a_1 \in A$
such that $a_0 > a_1$. Since $a_1$ is not a minimum element of $A$, there exists $a_2 \in A$ such that $a_1 > a_2$. Continuing this construction inductively, we find an infinite descending chain in $A$.

(d) Prove that if $(S, \leq)$ is a well-ordered set, then there exists a structure $(T, \preceq)$ such that $(S, \leq) \equiv (T, \preceq)$ and $(T, \preceq)$ is not a well-ordered set.

[Hint: Use part (c) and compactness.]

**Solution.** Let $M = (S, \leq)$ be a well-ordered set. Let $T = \{ \varphi \mid M \models \varphi \}$, and let

$$
\Gamma = T \cup \{ c_0 > c_1, c_1 > c_2, c_2 > c_3, \ldots \},
$$

where $c_0, c_1, c_2, \ldots$ are new constants. Any finite subset of $T$ is easily seen to be satisfiable by interpreting the the constant symbols $c_n$ adequately in $M$. Thus, by the Compactness Theorem, $\Gamma$ has a model, say $M^*$. Since $M^* \models T$, we have $M \equiv M^*$. In particular, $M$ is linearly ordered. However, $M$ has an infinitely decreasing chain, so it is not well-ordered, by part (c) above.

(e) Prove that the clause “the universe is well-ordered by $\leq$” cannot be written in first-order; that is, prove that there does not exist a first-order sentence $\varphi$ such that $(S, \leq) \models \varphi$ if and only if $\leq$ well-orders $S$.

**Solution.** Suppose that such a sentence exists. Then, if $(S, \leq)$ and $(S^*, \leq^*)$ are elementarily equivalent linear ordered and one of the structures is well-ordered, the other one is well-ordered as well. However, part (c) shows how to construct two elementarily equivalent linear orders such that one of them is well-ordered and the other one is not.

2 If $M$ is a structure in the language of arithmetic with universe $M$, an element $a$ of $M$ is said to be *finite* if there exists a natural number $n$ such that $a \leq \bar{n}$, where, as in class, $\bar{n}$ denotes the term

$$
\underbrace{1 + 1 + \ldots + 1}_n \text{ times}.
$$

An element of $M$ that is not finite is said to be *infinite*.

(a) Prove that there is no first-order formula $\varphi(x)$ in the language of arithmetic such that $M \models \varphi[a]$ if and only if $a$ is finite.
Solution. Suppose that such a formula \( \varphi(x) \) exists. Let \( N \) be the standard model of arithmetic and let \( M \) be a nonstandard model of arithmetic. Then, \( M \equiv N \), but \( N \models \forall x \varphi(x) \) and \( M \not\models \forall x \varphi(x) \), which is a contradiction.

(b) Prove that if \( M \) is a model of arithmetic with universe \( M \), then for every infinite \( a \in M \) there exists an infinite \( b < a \) in \( M \) such that \( a - b \) is infinite. [Note: Here, as in class, \( a - b \) denotes the unique \( c \in M \) such that \( b + c = a \).]

Solution. There are many ways to prove this. Here is one example. Fix a nonstandard model \( M \) of arithmetic with universe \( M \). Since \( M \) is elementarily equivalent to the standard model, \( M \models \forall x \exists y [2y \leq x \land x < 2y + 1] \).

Let \( a \) be an infinite element of \( M \) and let \( b \) be such that \( 2b \leq a < 2b + 1 \). It is easy to see that if \( a \) is larger than every finite element of \( M \), so is \( b \). Moreover, for every finite \( n \), we have \( b + n < a \); hence \( a - b \) is not finite.

3 (For 21-600) An existential sentence is a sentence of the form

\[
\exists x_1 \ldots \exists x_n \varphi(x_1, \ldots, x_n),
\]

where \( \varphi(x_1, \ldots, x_n) \) is a quantifier-free formula.

Let \( S \) be a signature and let \( M \) and \( N \) be \( S \)-structures. Prove that the following conditions are equivalent:

(i) Every existential sentence that is satisfied by \( M \) is satisfied by \( N \) as well.

(ii) There exists a structure \( N' \) such that \( N' \equiv N \) and \( N' \) contains a substructure isomorphic to \( M \).

[Hint: Use problem (4) of Assignment 6.]

Solution. Proof of (i)\(\Rightarrow\)(ii): Extend the signature by adding a new constant symbol \( c_a \) for each element \( a \) in the universe of \( M \). Let \( T = \{ \varphi \mid M \models \varphi \} \) and, as in (4) of Assignment 6, let \( \Delta_M \) be the set of all sentences of the form \( \psi(c_{a_1}, \ldots, c_{a_n}) \), where \( \psi(x_1, \ldots, x_n) \) is a quantifier-free \( S \)-formula. By (4) of Assignment 6, all we have to do is to show that \( T \cup \Delta_M \) has a model. Thus, by the Compactness Theorem, it suffices to show that every finite subset of \( T \cup \Delta_M \) has a model.

Fix a finite \( \Gamma \subseteq T \cup \Delta_M \). Then there exist \( \varphi_1, \ldots, \varphi_k \in T \), constant symbols \( c_{a_1}, \ldots, c_{a_n} \), and quantifier-free \( S \)-formulas
\(\psi_1(x_1, \ldots, x_n), \ldots, \psi_l(x_1, \ldots, x_n)\) such that

\[
\Gamma = \{ \varphi_1, \ldots, \varphi_k \} \cup \{ \psi_1(c_{a_1}, \ldots, c_{a_n}), \ldots, \psi(c_{a_1}, \ldots, c_{a_n}) \}.
\]

By the definition of \(T\) and \(\Delta_M\),

\[
M \models \bigwedge_{1 \leq i \leq k} \varphi_i \land \exists x_1 \ldots \exists x_n \left( \bigwedge_{1 \leq j \leq l} \psi_j(x_1, \ldots, x_n) \right),
\]

so, by (i), there exist elements \(b_1, \ldots, b_n\) of the universe of \(M\) such that

\[
M \models \bigwedge_{1 \leq i \leq k} \varphi_i \land \bigwedge_{1 \leq j \leq l} \psi_j[b_1, \ldots, b_n].
\]

By interpreting \(c_{a_1}, \ldots, c_{a_n}\) as \(b_1, \ldots, b_n\), respectively, we conclude that \(\Gamma\) has a model. This proves \((i) \Rightarrow (ii)\).

**Proof of \((ii) \Rightarrow (i)\):** Let \(N'\) be as given by (ii), and let \(N'_0\) be a substructure of \(N'\) isomorphic to \(M\). Fix a quantifier-free \(S\)-formula \(\psi(x_1, \ldots, x_n)\). If

\[
M \models \exists x_1 \ldots \exists x_n \psi(x_1, \ldots, x_n),
\]

then

\[
N'_0 \models \exists x_1 \ldots \exists x_n \psi(x_1, \ldots, x_n),
\]

and hence, since \(\psi(x_1, \ldots, x_n)\) is quantifier-free, it follows by straightforward induction on the complexity of \(\psi\) that

\[
N' \models \exists x_1 \ldots \exists x_n \psi(x_1, \ldots, x_n).
\]

Since \(\psi(x_1, \ldots, x_n)\) is arbitrary, this proves \((ii) \Rightarrow (i)\).