# Variational Aspects of Steady Irrotational Water Wave Theory

\- the fascination of what's difficult[1] -

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[1] a poem by W.B. Yeats (1916)

Water-Waves were studied Mathematically by

Cauchy, Laplace, Lagrange, Poisson, Green, Airy, Stokes, Rayleigh ...

and in 1845 Samuel Earnshaw complained that, a hundred years after the partial differential equations of fluid motion were given to the world by Euler, he was not aware of any case of fluid motion which had been rigorously extracted from them except in very simple and uninteresting cases. He said this was not from want of effort but from

"the peculiarly rebellious character of
the equations themselves which resist every attack"

Alex Craik: *George Gabriel Stokes on water wave theory.* Annual Review of Fluid Mechanics. **37** (2005), 23-42.

Note: Water with particles moving in parallel lines with constant speed under a horizontal surface is a "simple but uninteresting" water wave

Moral: It is not sufficient to prove that water-wave equations have solutions - we know that already! - they must be proved to be non-trivial

This is not always easy

After John Scott Russell (famously on horseback) discovered the solitary wave in 1834 and after ten years of laboratory experiments, both Stokes and Airy said he must be wrong because his published findings in 1844 did not agree with theory.

In this ground-breaking paper John Scott Russell described

*"the greater part of the investigations of Poisson and Cauchy under the name of wave theory are rather to be regarded as mathematical exercises, not physical investigations"*

and on the solitary wave (which he called the great wave of elevation) said

*... since no one had predicted the wave, it remained for mathematicians to give an "a priori demonstration a posteriori"*

Throughout the 19th century research focused on approximations in certain scaling limits Boussinesq (1877) and Koretweg & de Vries (1895), but the full equations remained essentially unexamined for more than one hundred years. However in the $20^{th}$ century significant advances were made.

## What is the Problem?

- a heavy liquid with an unknown free surface - gravity $g$ acts vertically down - infinite depth
- no viscosity - no surface tension - no floating bodies - 2D-irrotational flow -

In Eulerian coordinates the velocity at $(x, y, z)$ in the fluid at time $t$ is the gradient of a scalar velocity potential $\phi$ on $\mathbb{R}^2$

$$\vec{v}(x, y, z; t) = \nabla\phi(x, y; t)$$

which satisfies

$$\text{Wave Interior:} \quad \Omega = \{(x, y) : y < \eta(x, t)\}$$

$$\Delta\phi(x, y; t) = 0 \text{ on } \Omega$$

$$\text{Wave Surface} \quad \mathcal{S} = \{(x, \eta(x, t)) : x \in \mathbb{R}\}$$

$$\text{Boundary Condition} \quad \left.\begin{array}{r} \text{dynamic } \phi_t + \frac{1}{2}|\nabla\phi|^2 + gy = 0 \\ \text{kinematic } \eta_t + \phi_x\eta_x - \phi_y = 0 \end{array}\right\} \text{ on } \mathcal{S}$$

$$\text{At Infinite Depth} \quad \nabla\phi(x, y; t) \to 0 \text{ as } y \to -\infty$$

The challenge is to identify and classify solutions of all types (steady, travelling, standing, periodic, solitary etc) of all amplitudes

What follows is about waves that are $2\pi$-periodic in space, mainly (but not always) travelling with constant speed without changing shape

## A Special Case - Stokes Waves - Steady Periodic Water Waves

### with wavelength $\Lambda$ travelling waves with speed $c$ under gravity $g$

The stream function $\psi$, which is the harmonic conjugate of the velocity potential $\psi$, corresponding to a steady wave of period $\Lambda$, propagating without change of form with speed $c$, on an infinitely deep flow with gravity $g$, in a frame moving with the speed of the wave satisfies:

$\Delta\psi = 0$ in $\mathcal{S}$ ;

$\psi > 0$ in $\Omega$, $\quad \psi = 0$ on $\mathcal{S}$ (the surface $\mathcal{S}$ is a streamline)

$|\nabla\psi|^2 + 2\lambda y = 1$ on $\mathcal{S}$; on $\mathcal{S}$; (Bernoulli boundary condition)
equivalently $|\psi_\nu| = \sqrt{1 - 2\lambda y}$ on $\mathcal{S}$

$\nabla\psi(x, y) \to (0, -\lambda)$ as $y \to -\infty$,

where $\lambda$ is a dimensionless parameter incorporating $\Lambda$, $c$ and $g$.

The holomorphic (complex analytic) function $\phi + i\psi$ leads to a conformal mapping of one period of the flow domain $\Omega$ onto the unit disc in $\mathbb{C}$ which maps one period of $\mathcal{S}$ onto the unit circle.

This observation led to $\cdots\cdots\cdots$

## The First Proof of Existence of Non-Trivial Travelling Waves

Almost exactly 100 years ago, ***A. V. Nekrasov (1921)*** introduced his equation

$$\theta(s) = \frac{1}{3\pi} \int_{-\pi}^{\pi} \left( \sum_{k=1}^{\infty} \frac{\sin ks \sin kt}{k} \right) \frac{\sin \theta(t)}{\nu + \int_0^t \sin \theta(\nu) d\nu}$$

the slope $\theta : \mathbb{R} \to \mathbb{R}$ of $\mathcal{S}$ is $2\pi$ periodic, $\qquad \nu = \dfrac{3g\lambda c}{2\pi Q^3}$

*On steady waves*, Izv. Ivanovo-Voznesensk. Politekhn. In-ta **3** (1921)

and proved existence non-trivial two-dimensional steady periodic waves travelling without changing form on the surface of water which is inviscid, moving irrotationally, and at rest at infinite depth. For history, see

N G Kuznetsov *A tale of two Nekrasov equations* arXiv:2009.01754 (2020)

Although the equation is valid for waves of all amplitudes, his proof is of **small amplitude waves bifurcating from $\theta = 0$ at $\nu = 1$.**
In 1925 the celebrated geometer Tullio Levi-Civita developed a variant of Nekrasov's equation and found a different proof of the same result.

Both used complex variable, as noted earlier, to formulate the problem as an integral equation on the fixed domain $\mathbb{S}^1$

## Breakthrough: Large Amplitude Waves *Yu. P. Krasovskii* 1961

*On the theory of steady waves of finite amplitude*,
U.S.S.R. Comput. Math. Math. Phys. **1** (1961), 996–1018.

Using levi-Civita's version of Nekrasov's equation and subtle (for the time) global estimates in harmonic analysis, Krasovskii applied seminal work by Krasnosel'skii on Leray-Schauder degree theory to obtain,

$$\text{for all } \alpha \in [0, \pi/6), \text{ a solution } \theta \text{ with } \max_{[0,2\pi]} |\theta| = \alpha$$

***Strong arguments in hydrodynamic and harmonic analysis***
led experts to speculate that $\alpha \in [0, \pi/6)$ would be sharp.

In fact they were wrong:

McLeod (1979)[2] showed that $\max \alpha > \pi/6$ and

Amick (1987) showed that it was close!

$$\max \alpha < (1.098)\frac{\pi}{6}$$

But did this important result help address John Scott Russell's challenge to explain, or better predict, laboratory experiments, or the outcomes of numerical investigations, for large amplitude waves?

[2] MRC Tech. Report **2041**; published in 1997 *Stud. Appl. Math.*

## Available Methods

Most modern methods have been applied to the water-wave problem, but not with equal success

### Small Amplitudes

Implicit Function theory

Bifurcation Theory (Krasnosel'skii, Crandall & Rabinowitz)

Nash-Moser Theory (for Standing Waves) (Plotnikov, Iooss)

### Global Theory: Topological and Analytical

Topological Degree Theory (Krasnoselskii, Krasovskii, Crandall, Rabinowitz)

Global Bifurcation Theory (Rabinowitz, Dancer)

Global Real Analytic Function Theory (Dancer, Buffoni)

But Saddle-Point Mini-Max Principles, Mountain-Pass Lemma, Morse Index, Lyusternik-Schnirelman Category etc have been less successful
Nekrasov's equation is not in gradient form and $C[-\pi,\pi]$ is not reflexive

But even with a variational formulation there has been little progress -
- however there has been some - but much more needs to be done -

The rest of the lecture is about

## Variational Arguments in Water-Wave Theory

Energetics: The total wave energy at time $t$ is Kinetic + Potential

$$\frac{1}{2}\int\int_{-\infty}^{\eta(x,t)} |\nabla\phi(x,y;t)|^2| dydx + \frac{g}{2}\int \eta^2(x;t)dx$$

Given periodic functions $\eta$ and $\Phi$ of the single variable $x$ let

$$\Omega = \{(x,y) : y < \eta(x)\}$$

and let $\phi$ solve the corresponding Dirichlet problem

$$\left.\begin{array}{c}\Delta\phi(x,y) = 0 \\ \phi(x,y) \to 0 \text{ as } y \to -\infty\end{array}\right\} \text{ on } \Omega$$
$$\phi(x,\eta(x)) = \Phi(x), \quad x \in \mathbb{R}.$$

Then let

$$\mathcal{E}(\eta,\Phi) = \frac{1}{2}\int\int_{-\infty}^{\eta(x)} |\nabla\phi(x,y)|^2 dydx + \frac{g}{2}\int \eta^2(x)dx$$

The energy $\mathcal{E}$ is a function of the elevation $\eta$ of the free surface and the trace $\Phi$ of the velocity potential on it.

With this functional and with formal variational differentiation

$$\frac{\partial \mathcal{E}}{\partial \Phi} \qquad \text{and} \qquad \frac{\partial \mathcal{E}}{\partial \eta}$$

Zakharov (1968) observed that solutions $(\eta, \Phi)$ of the Hamiltonian system

$$\boxed{\frac{\partial \eta}{\partial t} = \frac{\partial \mathcal{E}}{\partial \Phi}(\eta, \Phi); \qquad \frac{\partial \Phi}{\partial t} = -\frac{\partial \mathcal{E}}{\partial \eta}(\eta, \Phi)}$$

yield solutions of the full time-dependent water wave equations

Benjamin & Olver (1982) treated this as an infinite-dimensional system:

$$\dot{x} = J \nabla \mathcal{E}(x), \quad x = (\eta, \Phi), \quad J = \begin{pmatrix} 0, I \\ -I, 0 \end{pmatrix},$$

$\eta$, $\phi$ being referred to as "coordinates" and "momentum". In an Appendix they gave the Hamiltonian formulation independent of canonical variables.

Both Zakharov and Benjamin knew the possible implications for stability of conservation laws in a variational setting such as this Hamiltonian system

However analysis of $\mathcal{E}$ is difficult because it involves the Dirichlet-Neumann operator on variable domains Craig & Sulem (1993), Lannes (2005).

# Time-Dependent Spatially Periodic Waves – normalised period $2\pi$

Any rectifiable $2\pi$-periodic Jordan curve $\mathcal{S} = \{(x, \eta(x)) : x \in \mathbb{R}\}$ in the plane can be parameterised as

$$\mathcal{S} = \{(-\xi - \mathcal{C}w(\xi), w(\xi)) : \xi \in \mathbb{R}\}$$

where $\mathcal{C}w$ is the Hilbert transform of a periodic function $w$:

$$\mathcal{C}w(\xi) = pv\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{w(\sigma)\,d\sigma}{\tan\frac{1}{2}(\xi-\sigma)}$$

This reduces Zakharov's awkward Hamiltonian system to the following "simpler" system:

$$\dot{w}(1+\mathcal{C}w') - \mathcal{C}\varphi' - w'\mathcal{C}\dot{w} = 0$$

$$\mathcal{C}\big(w'\dot{\varphi} - \dot{w}\varphi' + \lambda w w'\big) + (\dot{\varphi} + \lambda w)(1+\mathcal{C}w') - \varphi'\mathcal{C}\dot{w} = 0$$

$$\dot{} = \partial/\partial t, \qquad {}' = \partial/\partial x$$

$w$ = wave height $\varphi$ = potential at surface:

$0 < \lambda$ = gravity (dimensionless after normalisation), the wavelength is $2\pi$

Dyachenko, Kuznetsov, Spector & Zakharov (1996).

It does not look like ***a Hamiltonian system*** any more – ***but it is!***

## A Symplectic Form

For $(w,\varphi) \in M := W^{1,2}_{2\pi} \times W^{1,2}_{2\pi}$ let

$$\omega_{(w,\varphi)}\big((w_1,\varphi_1),(w_2,\varphi_2)\big) = \int_{-\pi}^{\pi} (1+\mathcal{C}w')(\varphi_2 w_1 - \varphi_1 w_2) + w'\big(\varphi_1 \mathcal{C} w_2 - \varphi_2 \mathcal{C} w_1\big) - \varphi'\big(w_1 \mathcal{C} w_2 - w_2 \mathcal{C} w_1\big)\, d\xi$$

This skew-symmetric bilinear form is exact (and so closed) because

$$\omega = d\varpi \quad \text{where} \quad \varpi_{\varphi,w}(\hat{w},\hat{\varphi}) = \int_{-\pi}^{\pi} \big\{\varphi(1+\mathcal{C}w') + \mathcal{C}\big(\varphi w'\big)\big\}\, \hat{w} d\xi$$

and by Riemann-Hilbert theory it is non-degenerate

## Hamilton System

$$\text{Hamiltonian } \mathcal{E}(w,\varphi) = \frac{1}{2}\int_{-\pi}^{\pi} \varphi\mathcal{C}\varphi' + \lambda w^2(1+\mathcal{C}w')\,d\xi$$

with the skew form $\omega$

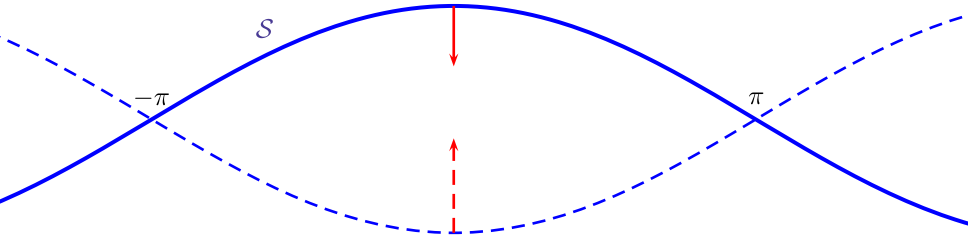
$$\begin{aligned}\omega_{(w,\varphi)}\big((\hat{w}_1,\hat{\varphi}_1),(\hat{w}_2,\hat{\varphi}_2)\big) \\ = \int_{-\pi}^{\pi} (1+\mathcal{C}w')(\hat{\varphi}_2\hat{w}_1 - \hat{\varphi}_1\hat{w}_2) \\ + w'\big(\hat{\varphi}_1\mathcal{C}\hat{w}_2 - \hat{\varphi}_2\mathcal{C}\hat{w}_1\big) \\ - \varphi'\big(\hat{w}_1\mathcal{C}\hat{w}_2 - \hat{w}_2\mathcal{C}\hat{w}_1\big)\,d\xi\end{aligned}$$

for $x$-periodic functions $(\phi(x,t), w(x,t))$ of real variables yields the Hamiltonian system

$$\boxed{\begin{aligned}\dot{w}(1+\mathcal{C}w') - \mathcal{C}\varphi' - w'\mathcal{C}\dot{w} = 0 \\ \mathcal{C}\big(w'\dot{\varphi} - \dot{w}\varphi' + \lambda w w'\big) + (\dot{\varphi}+\lambda w)(1+\mathcal{C}w') - \varphi'\mathcal{C}\dot{w} = 0\end{aligned}}$$

which is a very tidy version of the full time-dependent water-wave problem two equations, two real functions, one space and one time variable, fixed domain, quadratic nonlinearities,

## Brief Diversion: Standing Waves - Periodic in Space in Time




Siméon Denis Poisson (1781–1840) was first to think about standing waves – “le clapotis” (“lapping waves”) he called them –

This is a good example of how the Hamiltonian formulation helps organise a difficult problem by adapting Nash-Moser theory in the Hamiltonian theory of celestial mechanics

The problem with the failure of the classical Implicit Function Theorem is best explained in the physical plane

## The Difficulty is Clear from the Linearized Problem

## in the physical domain - too many solutions!

Standing Waves have normalised spatial period $2\pi$ and temporal period $T$

The velocity potential $\phi$ on the lower half plane $\{(x,t) :\in \mathbb{R}^2 : y < 0\}$:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \qquad x,\ t \in \mathbb{R}, \quad y < 0,$$

Boundary Conditions

$$\phi(x + 2\pi, y; t) = \phi(x, y; t) = \phi(x, y; t + T), \qquad x,\ t \in \mathbb{R}, \quad y < 0,$$

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial y} = 0, \qquad y = 0$$

$$\phi(-x, y; t) = \phi(x, y; t) = -\phi(x, y; -t), \qquad x,\ t \in \mathbb{R}, \quad y < 0$$

$$\nabla\phi(x, y; t) \to (0, 0), \qquad y \to -\infty$$

The wave Elevation $\eta$:

$$g\eta(x,t) = -\frac{\partial \phi}{\partial t}(x, 0, t)$$

In 1818 Poisson observed that when $\lambda := gT^2/2\pi$ is irrational there are no non-constant solutions

However, when $\lambda \in \mathbb{Q}$, for every $m$, $n$ with $\frac{n^2}{m} = \lambda$

$$\phi(x, y, t) = \sin\left(\frac{2n\pi t}{T}\right)\cos\Big(2m\pi x\Big)\exp\Big(2m\pi y\Big)$$

is a solution

The eigenvalues $\lambda$ of the linearized problem are $\mathbb{Q} \cap [0, \infty)$ which is dense and each eigenspace is infinite-dimensional

This is called complete resonance. It is a huge nightmare because when attacking the nonlinear problem inversion of the linearised problem involves small-divisor problems.

However following work by Amick (1984) and others, in the past decade this Hamiltonian formulation combined with the Nash-Moser approach (from differential geometry) has led to non-trivial small amplitude solutions of the full nonlinear problem for a measurable set of $\lambda$ which is dense at 1 Plotnikov (2001), Plotnikov & Iooss(2004, 2005)

End of Brief Diversion

## Periodic Travelling Waves (Stokes Waves): Babenko's Equation

With $\phi(x,t) = \phi(x - ct)$ and $w(x,t) = w(x - ct)$ the Hamiltonian system simplifies dramatically to $\phi' = cw'$ and an equation for $w$ only:

$$\mathcal{C}w' = \lambda\big(w + w\mathcal{C}w' + \mathcal{C}(ww')\big) \qquad (*)$$

Here the wave speed $c$ has been absorbed in $\lambda$, the Froude number squared

An Aside on Duality: With $v := \lambda w^2 - w$, equation $(*)$ can be rewritten

$$v'(1 + \mathcal{C}w') + w'(1 + \mathcal{C}v') = 0$$

which means that $v$ satisfies $(*)$ and so corresponds to another free boundary problem on infinite depth, namely

$$\Delta\hat{\psi} = 0 \text{ in } \hat{\Omega}, \quad \hat{\psi} = 0 \text{ and } (4\lambda y + 1)|\hat{\psi}_\nu|^4 \equiv 1 \text{ on } \hat{\mathcal{S}} \qquad (\dagger)$$

This is "dual" to the classical Stokes wave free boundary problem for $\psi$, the steady stream function satisfies

$$\Delta\psi = 0 \text{ in } \Omega, \quad \psi = 0 \text{ and } |\psi_\nu| = \sqrt{1 - 2\lambda y} \text{ on } \mathcal{S} \qquad (\ddagger)$$

Although $(\dagger)$ and $(\ddagger)$ are equivalent, $\quad \Omega \neq \hat{\Omega}$ and $\mathcal{S} \neq \hat{\mathcal{S}}$

Thus any solution of Nekrasov's equation, or of Babenko's equation, yields simultaneously solutions to two distinct Bernoulli free boundary problems

## Variational Structure of Babenko's Equation

$$\mathcal{C}w' = \lambda\big(w + w\mathcal{C}w' + \mathcal{C}(ww')\big) \qquad (*)$$

Note that $w \mapsto \mathcal{C}w'$ is first-order, non-negative-definite, self-adjoint which is densely defined on $L^2_{2\pi}$ by $\mathcal{C}(e^{ik})' = |k|e^{ik}, \quad k \in \mathbb{Z}$. Hence

$$\mathcal{C}w' = \sqrt{-\frac{\partial^2}{\partial\xi^2}}\, w$$

behaves like an elliptic differential operator but lacks a maximum principle and the $H^{1/2}(\mathbb{S}^1)$ norm of $w$ is given by

$$\|w\|^2_{H^{1/2}(\mathbb{S}^1)} = \|w\|^2_{L_2(\mathbb{S}^1)} + \int_0^{2\pi} w\mathcal{C}w'\, dt$$

Note also that $(*)$ is the Euler-Lagrange equation of

$$\int_{-\pi}^{\pi} w\mathcal{C}w' - \lambda\big(w^2(1 + \mathcal{C}w')\big)\, d\xi, \qquad (\star\star)$$

a simple functional on $W^{1,2}(\mathbb{S}^1)$, but unfortunately not on $H^{1/2}(\mathbb{S}^1)$.

So there is no ***self-contained variational proof*** of existence of critical points of $(\star\star)$ that yield large-amplitude solutions of $(*)$

But variational methods are effective in alliance with other methods

## The Stokes Wave Equation

Equation $(*)$ is cubic in $w$ and can be re-written

$$(1 - 2\lambda w)\mathcal{C}w' = \lambda(w - (w\mathcal{C}w' - \mathcal{C}(ww'))$$

where by magic

$$w\mathcal{C}w' - \mathcal{C}(ww') = \frac{1}{8\pi}\int_0^{2\pi}\left\{\frac{w(x) - w(y)}{\sin\frac{1}{2}(x-y)}\right\}^2 dy \geqslant 0,$$

and is in $L_q$ for $q > p > 1$ when $w' \in L_p$. Difficulty arises only when $1 - 2\lambda w$ has zeros, which corresponds to Stokes wave of extreme form (of which more later)

When $1 - 2\lambda w > 0$ $(*)$ can be written

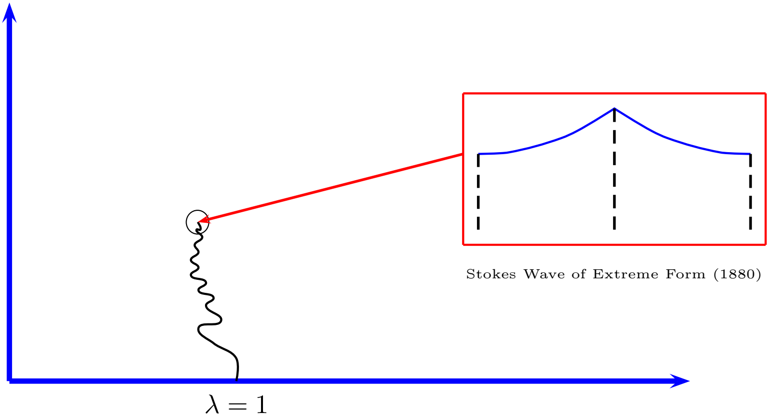
$$\mathcal{C}w' = \lambda\left(\frac{w + \mathcal{C}(ww') - w\mathcal{C}w'}{1 - 2\lambda w}\right)$$

This is a friendly formulation the problem that vexed Lord Rayleigh and was finally settled by Nekrasov and Levi Civita in the 1920s.

Nowadays a very elementary application of bifurcation from a simple eigenvalue yields their small-amplitude waves with $\lambda$ close to 1.

Global bifurcation theory has much more to say, but many global questions remain unanswered

Numerical Evidence Suggests a Global Bifurcation Picture Like This:




Numerically:
energy oscillates
maximum slope oscillates
number of inflection points increases without bound
as the extreme wave $(1 - 2\lambda w = 0)$ is approached

## Morse Index $\mathcal{M}(w)$

The Morse index $\mathcal{M}(w)$ of a critical point $w$ is the number of negative eigenvalues $\mu < 0$ of $D^2\mathcal{J}(w)$, where $D^2\mathcal{J}[w]$ is the linearization of $(*)$ at $(\lambda, w)$:

$$D^2\mathcal{J}(w)\,\phi = \mathcal{C}\phi' - \lambda\big(\phi + \phi\mathcal{C}w' + w\mathcal{C}\phi' + \mathcal{C}(w\phi)'\big)$$

The Morse Index may be infinite if $1 - 2\lambda w$ has zeros (at an extreme wave)

### Plotnikov's Theorem

Suppose a sequence $\{(\lambda_k, w_k)\}$ of solutions of $(*)$ has $1 - 2\lambda_k w_k \neq 0$ and the Morse indices $\{\mathcal{M}(w_k)\}$ are bounded. Then for some $\alpha > 0$

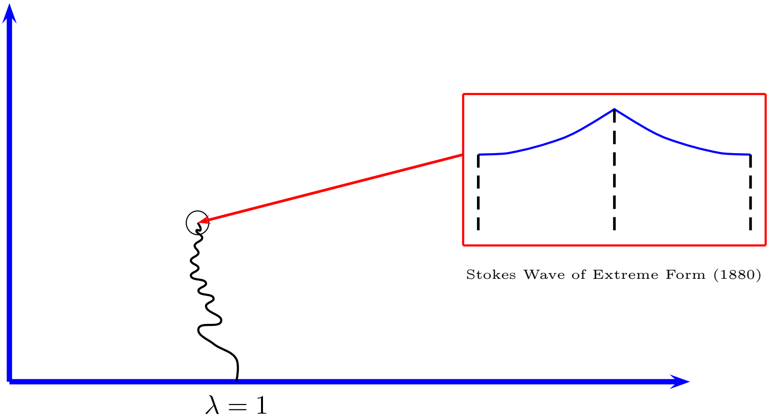
$$\boxed{1 - 2\lambda_k w_k(x) \geqslant \alpha, \quad x \in \mathbb{R}, \quad k \in \mathbb{N}}$$

Solutions with bounded Morse index are bounded away from extreme waves

Alternatively, Stokes waves approaching extreme form becomes more and more unstable in the sense of Morse indices

Shargorodsky (2013) quantified the relation between the Morse index and $\alpha$

Primary Branch




Moreover as the extreme wave is approached:
energy oscillates
maximum slope oscillates
number of inflection points increases without bound
Morse index grows without bound

## Plotnikov's Result Means More than Appears

In abstract terms Babenko's equation for travelling waves is

$$\mathcal{C}w' = \lambda \nabla \varPhi(w) \tag{‡‡}$$

where

$$\varPhi = \frac{1}{2}\int_{-\pi}^{\pi} \left\{ w^2(1 + \mathcal{C}w') \right\} \, dx$$

By real-analytic function theory there is a parameterized real-analytic curve of solutions $\{(\lambda_s, w_s) : s \in [0, \infty)\}$ with $\mathcal{M}(w_s) \to \infty$ as $s \to \infty$

Suppose that the Morse Index changes as $s$ passes through $s^*$

Then

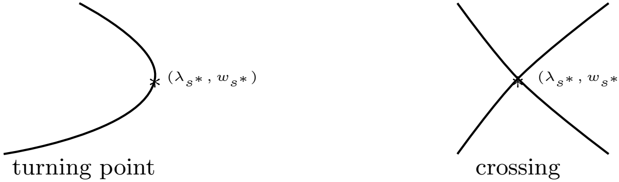
- The *variational structure* of (‡‡) with $\lambda$ a multiplier on the right

  and

- the fact that the Morse Index changes as $s$ passes through $s^*$

  mean

# One of Two Things Happens




The numerical evidence is that the first happens each time the Morse index changes

but there is no proof

It seems to be a very hard problem

Motivated by numerics of Chen & Saffman (1980) there's more can be said if things are looked at in a slightly different setting

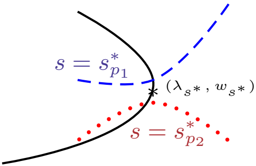
# A Slightly Different Setting

The solutions $\{(\lambda_s, w_s) : s \in [0,\infty)\}$ with $\mathcal{M}(w_s) \to \infty$ as $s \to \infty$ are $2\pi$-periodic and hence $2p\pi$-periodic for any prime number $p$

Let $\mathcal{M}_p(w)$ denote their Morse index in that new setting

For $p$ sufficiently large and prime, it can be shown that $\mathcal{M}_p(w_s)$ changes as $s$ passes through $s_p^*$ when $s_p^*$ is close to $s^*$ but $s_p^* \neq s^*$

Thus near $s^*$ on the primary branch there is a bifurcation point $s_p^*$ for solutions of minimal period $2p\pi$




Period-multiplying (sub-harmonic) bifurcation near turning points on the primary branch was observed numerically by Chen & Saffman (1980)

Alas, mathematicians again discovered an a priori proof a postiori

## Open Question

By methods of topological-degree and real-analytic methods the global branch of Stokes waves “terminates” at Stokes extreme wave.

Plotnikov’s result guarantees solutions with arbitrarily large Morse index

Despite the simple form of $\mathcal{J}$, a satisfactory ***global variational approach*** to the existence of Stokes waves remains undiscovered

Open questions abound: an obvious one is:

*For all large $n \in \mathbb{N}$ does there exist a wave with Morse index $n$?*

## Summary of Variational Approach to Babenko's Equation $(*)$

Equation $(*)$ for small-amplitude, steady waves was published by Babenko in 1987 - the year in which he died) and he noted its variational structure.

Independently Plotnikov (1992) and Balk (1996) rediscovered equation $(*)$

Dyachenko, Kuznetsov, Spector & Zakharov (1996), used conformal mappings to transform Zakharov's (1968) Hamiltonian system, but did not comment on the outcome as a Hamilton system in its own right. They derived $(*)$ for travelling waves, apparently unaware of Babenko (1987)

Plotnikov (1992) introduced Morse index calculations for an analogue of $(*)$ in his brilliant study of the non-uniqueness question for solitary waves

Buffoni & Séré (2003, 2005) and Buffoni & Dancer (1998, 2000) added to topological existence theory using the variational structure.

Buffoni (2001) uncovered the dual free boundaries for Stokes waves

In the last twenty years the theory of $(*)$ has been extended to cover a general class of free boundary problems Shargorodsky (2003, 2008, 2013) But nothing emerged to make Stokes Waves special in that wider class

**There remain many open mathematical questions !**

It's a good time to stop !!

**Thank You!!!**