These notes are from http://math.cmu.edu/~cocox/teaching/discrete20/rec7.pdf

We showed in lecture that $R(3,3) = 6$; that is, any red-blue coloring of the edges of K_6 must have a monochromatic triangle, and there is a red-blue coloring of the edges of K_5 which does not. We could extend this question further: given a positive integer n and a coloring $\chi\colon E(K_n) \to \{r, b\}$, how many monochromatic triangles must χ have? Let C_n denote the set of all red-blue colorings of $E(K_n)$, and for $\chi \in C_n$, let $T(\chi)$ denote the number of monochromatic triangles in χ.

$R(3,3) = 6$ means that for every $n \in [5]$, there is some $\chi \in C_n$ with $T(\chi) = 0$ and for every $n \geq 6$ and $\chi \in C_n$ we have $T(\chi) \geq 1$.

Of course, χ could simply color every edge red, so we could have $T(\chi) = \binom{n}{3}$, which is just the total number of triangles in K_n, but most colorings of $E(K_n)$ won't have anywhere near this many. We are interested in the number of monochromatic triangles that a red-blue coloring of $E(K_n)$ *must* have. Thus, define

$$T(n) := \min_{\chi \in C_n} T(\chi).$$

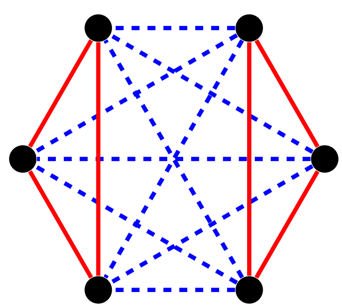
Since $R(3,3) = 6$, we can show that $T(n) \geq \lfloor n/6 \rfloor$. Indeed, we can break up K_n into $\lfloor n/6 \rfloor$ disjoint K_6's, and each of these will give us at least one monochromatic triangle. In fact, roughly the same idea works to show that $T(n) \geq \left\lfloor \frac{n-2}{4} \right\rfloor$ (how?). By being less wasteful, we can use some of the same ideas to get $T(n) \geq \frac{n}{3} - c$ where c depends only on the value of n modulo 6 (how?).

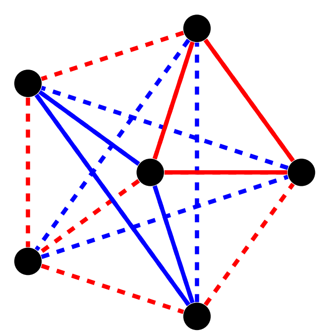
To reiterate, we know that $T(5) = 0$ and $T(6) \geq 1$. Interestingly:

Claim 1. $T(6) = 2$.

We'll differ the proof that $T(6) \geq 2$ for just a bit since we'll prove something more general. But here's a proof of the upper bound.

Proof Claim 1 (upper bound). We need to construct a red-blue coloring of $E(K_6)$ which has at most two monochromatic triangles. Two very different colorings which accomplish this are shown below. The solid edges form the monochromatic triangles; all other edges are dashed for readability.





□

Using Claim 1, we can get another proof that $T(n) \geq \frac{n}{3} - c$ where c depends only on the value of n modulo 6 (how?). However, it turns out that this lower bound is very far from the truth when n is large.

Claim 2. $T(n) \gtrsim \frac{1}{4}\binom{n}{3}$.

Here, $f(n) \gtrsim g(n)$ means that there is some $h(n)$ such that $f(n) \geq h(n)$ and $h(n) \sim g(n)$. Therefore, the claim states that, for n sufficiently large, in any red-blue coloring of $E(K_n)$, roughly a quarter of all triangles must be monochromatic. Additionally, it turns out that $T(n) \sim \frac{1}{4}\binom{n}{3}$ since there is a red-blue coloring of $E(K_n)$ in which at most a quarter of all triangles are monochromatic.[1]

Both Claims 1 and 2 will follow from the lemma below.

Lemma 3. *In any $\chi\colon E(K_n) \to \{r, b\}$, there are at most $n(n-1)^2/8$ non-monochromatic triangles.*

Before proving the lemma above, let's quickly use it to prove the claims.

Proof Claim 1 (lower bound). Fix any $\chi\colon E(K_6) \to \{r, b\}$. By Lemma 3, there are at most $6 \cdot 5^2/8 = 18.75$ non-monochromatic triangles in χ. Of course, the number of non-monochromatic triangles is an integer, so there are at most 18 of these. Since there are $\binom{6}{3} = 20$ total triangles in K_6, this means that χ must have at least two monochromatic triangles. □

Proof of Claim 2. Fix any $\chi\colon E(K_n) \to \{r, b\}$. By Lemma 3, the number of monochromatic triangles in χ is at least

$$\binom{n}{3} - \frac{n(n-1)^2}{8} \sim \frac{n^3}{6} - \frac{n^3}{8} = \frac{n^3}{24} \sim \frac{1}{4}\binom{n}{3}.$$ □

Proof of Lemma 3. Let N denote the set of all non-monochromatic triangles in χ. Define the set

$$P := \big\{(x, y, z) \in V(K_n)^3 : \chi(xy) = r \wedge \chi(yz) = b\big\}.$$

Observe first that if $(x, y, z) \in P$, then x, y, z are distinct vertices since every edge has only one color and there is not an edge from a vertex to itself. Next, observe that for $(x, y, z) \in P$, the triangle with vertices x, y, z is not monochromatic. Furthermore, every non-monochromatic triangle gives rise to exactly two unique elements of P. In particular, suppose that $\{x, y, z\} \in N$ with, say, $\chi(xy) = \chi(xz) = r$ and $\chi(yz) = b$; then $(x, y, z), (x, z, y) \in P$.

Putting these observations together, we have $|P| = 2|N|$, so it's enough to get an upper bound on $|P|$. We'll do so by partitioning P based on the second coordinate.

For a vertex $v \in V(K_n)$, let $\deg_r(v)$ denote the number of red edges incident to v and let $\deg_b(v)$ denote the number of blue edges incident to v. We compute

$$\begin{aligned} |P| &= \sum_{v \in V(K_n)} \big|\big\{(x, y) \in V(K_n)^2 : \chi(xv) = r \wedge \chi(vy) = b\big\}\big| \\ &= \sum_{v \in V(K_n)} \deg_r(v)\deg_b(v). \end{aligned}$$

[1] Consider flipping a fair coin to determine the color of each edge. Once we learn about linearity of expectation, it will be routine to show that the average number of monochromatic triangles in such a coloring is $\frac{1}{4}\binom{n}{3}$. Since not everyone can be above average, this means that there is some coloring χ with $T(\chi) \leq \frac{1}{4}\binom{n}{3}$.

Finally, $\deg_r(v)+\deg_b(v) = n-1$, so $\deg_r(v)\deg_b(v) \le \left(\frac{n-1}{2}\right)^2$ (since the maximum of $x(1-x)$ occurs at $x = 1/2$). Therefore,

$$2|N| = |P| \le \sum_{v\in V(K_n)} \left(\frac{n-1}{2}\right)^2 = \frac{n(n-1)^2}{4}. \qquad \square$$