These notes are from http://math.cmu.edu/~cocox/teaching/discrete20/rec4.pdf

Today we're going to look at another way to think about the inclusion-exclusion formula and work with signed sums. This technique, which I call the "matching method," mirrors the ideas from double counting. As is the norm in this class, the best way to teach this technique is through examples.

Claim 1. *For $k < n$, $\sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{i}{k} = 0$.*

Proof. This identity looks a lot like $\sum_{i=0}^{n}\binom{n}{i}\binom{i}{k} = 2^{n-k}\binom{n}{k}$, which we've proved by double counting the set

$$\Omega := \left\{(A,B) \in 2^{[n]} \times \binom{[n]}{k} : B \subseteq A\right\}.$$

Unfortunately, we have these pesky negatives... In this type of alternating sum, some pieces will cancel out and we're trying to figure out what's left over (in this problem, nothing should be). Thus, informally speaking, instead of trying to count Ω, we're trying to pair up elements of Ω so that they "cancel out."

Let's try to imitate a double counting proof, but throw in signs somewhere. For $(A,B) \in \Omega$, define its sign to be $\sigma(A,B) = (-1)^{|A|}$. Setting $Q := \sum_{(A,B)\in\Omega}\sigma(A,B)$, we have

$$Q = \sum_{i=0}^{n}\sum_{A\in\binom{[n]}{i}}(-1)^{|A|}\left|\left\{B \in \binom{[n]}{k} : B \subseteq A\right\}\right| = \sum_{i=0}^{k}(-1)^i\binom{n}{i}\binom{i}{k}.$$

When looking at Q, we can see that many of the σ's should cancel out (in fact, all of them theoretically should), so let's try to pair up (A,B)'s of opposite signs.

For $B \in \binom{[n]}{k}$, define $F(B) = \min([n] \setminus B)$, which is well-defined since $|B| = k < n$. Now, for $(A,B) \in \Omega$, define $f(A,B) = (A\triangle\{F(B)\}, B)$. Observe first that since $F(B) \notin B$, we have $f(A,B) \in \Omega$. Furthermore, since $\triangle$ is an involution, so is f: $f(f(A,B)) = (A,B)$. Therefore, f gives us a way to pair up these (A,B)'s! The last key observation is that f switches signs: since $|A|$ and $|A\triangle\{F(B)\}|$ differ by exactly one, $\sigma(A,B) = -\sigma(f(A,B))$. As such, when evaluating Q, (A,B) and $f(A,B)$ will cancel! We therefore have

$$Q = \sum_{\{(A,B),f(A,B)\}}\big(\sigma(A,B) + \sigma(f(A,B))\big) = 0. \qquad \square$$

We can use the matching method as an alternative in all sorts of inclusion-exclusion problems! Let's use it to count derangements. Let D_n be the set of derangements of $[n]$; that is permutations of $[n]$ with no fixed points.

Claim 2. $|D_n| = n! \cdot \sum_{k=0}^{n}\frac{(-1)^k}{k!}$.

Proof. To motivate the key insight in this proof, recall that there is a bijection between functions from $[m] \to [n]$ and words of length m whose letters come from $[n]$. Indeed, a function $f\colon [m] \to [n]$ can be represented by the word $f(1)f(2)\cdots f(m)$. In light of this bijection, a permutation of $[n]$ is a word of length n whose letters come from $[n]$ and has no repeated letter.

Let $[n]_k$ denote the set of k-letter words coming from $[n]$ so that no letter is repeated. Note that $|[n]_k| = \frac{n!}{(n-k)!}$, which looks very similar to the terms in the sum. Define

$$\Omega := \bigcup_{k=0}^{n} [n]_k,$$

which is the set of all words of length at most n whose letters come from $[n]$ and have no repeated letter. For a word $w \in \Omega$, we associate a sign defined by $\sigma(w) = (-1)^{n-k}$ where k is the length of w. Define $Q := \sum_{w\in\Omega} \sigma(w)$, which is the sum of all of these signs. Observe that

$$Q = \sum_{w\in\Omega} \sigma(w) = \sum_{k=0}^{n} \sum_{w\in[n]_k} \sigma(w) = \sum_{k=0}^{n} \sum_{w\in[n]_k} (-1)^{n-k} = \sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{(n-k)!} = n! \cdot \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

If we can show that $Q = |D_n|$, then we're done!

For $w \in \Omega$, define $F(w)$ to be the smallest $i \in [n]$ such that either i does not appear in w or i is fixed in w (that is, the i'th letter of w is i). However, $F(w)$ is not defined for every $w \in \Omega$. It turns out that $\{w \in \Omega : F(w) \text{ undefined}\} = D_n$ since these are precisely the non-repeating words of length at most n that have all n letters and no letter is fixed!

Consider a words w for which $F(w)$ is defined. If $F(w)$ does not appear in w, then define $f(w)$ to be the word formed by inserting $F(w)$ into its fixed position. If $F(w)$ is fixed in w, then define $f(w)$ to be the words formed by removing $F(w)$ from w. For example, $f(1354) = 354$ and $f(21456) = 213456$.

Observe that f is a map between words for which F is defined; in fact $F(w)$ is the same as $F(f(w))$. Therefore, f is actually an involution: $f(f(w)) = w$. In other words, f gives us a way to pair up words for which F is defined! The final key step is to observe that w and $f(w)$ always differ by one in length, so $\sigma(w) = -\sigma(f(w))$. Therefore, by pairing up w and $f(w)$ whenever possible, we have

$$\begin{aligned} Q &= \sum_{\substack{w: \\ F(w) \text{ undefined}}} \sigma(w) + \sum_{\{w, f(w)\}} \big(\sigma(w) + \sigma(f(w))\big) \\ &= \sum_{\substack{w: \\ F(w) \text{ undefined}}} \sigma(w) = \sum_{w\in D_n} \sigma(w) = \sum_{w\in D_n} (-1)^{n-n} = |D_n|. \end{aligned}$$

□

In fact, we can even prove the inclusion-exclusion formula by using this technique!

Claim 3. *Let Ω be a finite set and let $B_1, \ldots, B_n \subseteq \Omega$. Then*

$$\left|\Omega \setminus \bigcup_{i\in[n]} B_i\right| = \sum_{S\subseteq[n]} (-1)^{|S|} \left|\bigcap_{i\in S} B_i\right|.$$

Proof. Define the set

$$\mathcal{B} := \left\{ (x,S) \in \Omega \times 2^{[n]} : x \in \bigcap_{i \in S} B_i \right\}.$$

In other words, the elements of $\mathcal{B}$ consist of a point $x \in \Omega$ and an index set S so that x lives in all of the B_i's indexed by this S. Note that S here doesn't have to be the set of all of the indices for which $x \in B_i$; since $\bigcap_{i\in\varnothing} B_i = \Omega$, we have $(x, \varnothing) \in \mathcal{B}$ for all $x \in \Omega$, even if x does live in some of the B_i's.

For $(x,S) \in \mathcal{B}$, define its sign by $\sigma(x,S) = (-1)^{|S|}$. Setting $Q := \sum_{(x,S)\in\mathcal{B}} \sigma(x,S)$, we have

$$Q = \sum_{S \subseteq [n]} (-1)^{|S|} |\{x \in \Omega : (x,S) \in \mathcal{B}\}| = \sum_{S\subseteq[n]} (-1)^{|S|} \left| \bigcap_{i\in S} B_i \right|.$$

Now, we want to evaluate Q in a different way by canceling as many signs as possible. For $x \in \Omega$, define $F(x) = \min\{i \in [n] : x \in B_i\}$. Observe that $F(x)$ is undefined if and only if $x \in \Omega \setminus \bigcup_{i\in[n]} B_i$, which is what we'd like to count. Additionally, note that if $F(x)$ is undefined, then $(x,S) \in \mathcal{B}$ if and only if $S = \varnothing$.

Now, consider a pair $(x,S) \in \mathcal{B}$ where $F(x)$ is defined and let $f(x,S) = (x, S \triangle \{F(x)\})$. Whether or not $F(x) \in S$, since $x \in B_{F(x)}$ by definition, $f(x,S) \in \mathcal{B}$. Furthermore, since the symmetric difference is an involution, so is f! Finally, since $|S \triangle \{F(x)\}|$ and $|S|$ differ by exactly one, we have $\sigma(x,S) = -\sigma(f(x,S))$, so the signs will cancel when we pair up (x,S) and $f(x,S)$. From this, we compute

$$\begin{aligned} Q &= \sum_{\substack{(x,S)\in\mathcal{B}: \\ F(x) \text{ undefined}}} \sigma(x,S) + \sum_{\{(x,S), f(x,S)\}} \big(\sigma(x,S) + \sigma(f(x,S))\big) \\ &= \sum_{\substack{(x,S)\in\mathcal{B}: \\ F(x) \text{ undefined}}} \sigma(x,S) = \sum_{\substack{(x,S)\in\mathcal{B}: \\ F(x) \text{ undefined}}} (-1)^{|\varnothing|} = \left| \Omega \setminus \bigcup_{i\in[n]} B_i \right|. \end{aligned}$$

□

As a take-away, the matching method is an alternative way to think about inclusion-exclusion and to work with general signed sums. However, it can oftentimes be easier to just plug-and-chug through the inclusion-exclusion formula instead since that requires relatively little thought compared to these ideas. It's still a good idea to understand this method, though. Here are two more problems to try out for practice.

1. For $m \le n$, show that $\sum_{i=0}^{m} (-1)^i \binom{n}{i} = (-1)^m \binom{n-1}{m}$.

2. Show that number of surjections from $[m]$ to $[n]$ is $\sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^m$.

   - Hint: Recall that functions from $[m]$ to $[n]$ can be represented by words of length m whose letters come from $[n]$; therefore, surjections correspond to those words in which each letter appears at least once. Thus, consider working with the set $\big\{(X,Y) : X \subseteq [n] \ \wedge \ Y \in \big([n] \setminus X\big)^m\big\}$.

Remark [not discussed in recitation]. In Recitation 2, we mentioned that $|D_n| = \lfloor \frac{n!}{e} + \frac{1}{2} \rfloor$; that is, $|D_n|$ is the closest integer to $n!/e$. We can now prove this! As a first step, recall that $e^x = \sum_{k\geq 0} x^k/k!$. By applying the formula in Claim 2, we find that

$$\lim_{n\to\infty} \frac{|D_n|}{n!} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!} = \sum_{k\geq 0} \frac{(-1)^k}{k!} = e^{-1},$$

so $|D_n|$ is asymptotic to $n!/e$. Of course, this doesn't mean that $|D_n|$ is simply $n!/e$ rounded to the nearest integer since the error term here could be large. Therefore, we'll need to do a bit more work in order to show this.

Lemma 4. *For every $n \geq 1$, we have* $\left| \sum_{k\geq n} \frac{(-1)^k}{k!} \right| < \frac{1}{n!}$.

Proof. Observe first that

$$\left| \sum_{k\geq n} \frac{(-1)^k}{k!} \right| = \left| \sum_{k\geq n} \frac{(-1)^{k-n}}{k!} \right|,$$

so we can work instead with the latter sum. We first bound

$$\sum_{k\geq n} \frac{(-1)^{k-n}}{k!} = \sum_{k\geq 0} \left(\frac{1}{(n+2k)!} - \frac{1}{(n+2k+1)!} \right) = \sum_{k\geq 0} \frac{n+2k}{(n+2k+1)!} > 0.$$

On the other hand,

$$\sum_{k\geq n} \frac{(-1)^{k-n}}{k!} = \frac{1}{n!} + \sum_{k\geq 1} \left(-\frac{1}{(n+2k-1)!} + \frac{1}{(n+2k)!} \right) = \frac{1}{n!} - \sum_{k\geq 1} \frac{n+2k-1}{(n+2k)!} < \frac{1}{n!}. \qquad \square$$

Claim 5. *For $n \geq 1$, $|D_n|$ is the closest integer to $n!/e$.*

Proof. Since $|D_n|$ is an integer by definition, all we need to show is that $\left| \frac{n!}{e} - |D_n| \right| < \frac{1}{2}$. Applying Claim 2 and Lemma 4, we find that

$$\begin{aligned} \left| \frac{n!}{e} - |D_n| \right| &= \left| n! \cdot \sum_{k\geq 0} \frac{(-1)^k}{k!} - n! \cdot \sum_{k=0}^{n} \frac{(-1)^k}{k!} \right| = \left| n! \cdot \sum_{k\geq n+1} \frac{(-1)^k}{k!} \right| \\ &< \frac{n!}{(n+1)!} = \frac{1}{n+1} \leq \frac{1}{2}. \end{aligned} \qquad \square$$