In HW7(5), you were asked to show that if G is a graph with m edges, then G contains a bipartite graph with at least $m/2$ edges. Let's extend this result.

Claim 1. *Let $G_1, \dots, G_k$ be graphs on a common vertex set V, each with m edges. There is a partition of $V = A \sqcup B$ so that for each i, G_i has at least $\frac{m}{2} - \sqrt{km}$ edges between A and B.*

As a remark, each G_i has a biparition with at least $\frac{m}{2}$ crossing edges, but this bipartition need not be the same for each G_i. This result says that we can use the same bipartition for each graph if we're willing to give up just a little bit.

Proof. Independently for each $v \in V$, include v in A with probability $1/2$ and otherwise include v in B. This creates a random bipartition $V = A \sqcup B$.

Let $G = (V, E)$ be any graph with m edges. Let X be the random variable which counts the number of edges of G with one vertex in A and the other in B. Of course, $X = \sum_{e \in E} X_e$ where X_e is the indicator random variable of the event that edge e crosses between A and B.

In HW7(5), you verified that $\mathbb{E}\, X_e = \mathbf{Pr}[e \text{ crosses}] = \frac{1}{2}$, and so $\mathbb{E}\, X = \frac{m}{2}$. We now compute $\mathbf{Var}\, X$. For any $e, s \in E$, we find that

$$\begin{aligned}\mathbb{E}\, X_e X_s = \mathbf{Pr}[e \text{ and } s \text{ cross}] &= \mathbf{Pr}[e \text{ crosses} \mid s \text{ crosses}]\, \mathbf{Pr}[s \text{ crosses}] \\ &= \begin{cases} \frac{1}{2} & \text{if } e = s, \\ \frac{1}{4} & \text{if } e \neq s. \end{cases}\end{aligned}$$

Now, a bit of algebraic manipulation yields (do this for yourself!)

$$\mathbf{Var}\, X = \sum_{e,s \in E} \left(\mathbb{E}\, X_e X_s - \mathbb{E}\, X_e\, \mathbb{E}\, X_s\right) = \sum_{e \in E} \left(\frac{1}{2} - \frac{1}{4}\right) + \sum_{e \neq s \in E} \left(\frac{1}{4} - \frac{1}{4}\right) = \frac{m}{4}.$$

For each $i \in [k]$, let X_i denote the number of crossing edges of G_i, so since G_i has vertex set V and has m edges, the facts derived above for X hold also for X_i, i.e. $\mathbb{E}\, X_i = \frac{m}{2}$ and $\mathbf{Var}\, X_i = \frac{m}{4}$. We now combine the above results with the union bound and Chebyshev's inequality to prove the claim. Observe that

$$\begin{aligned}\mathbf{Pr}\left[\bigcup_{i \in [k]} \left\{X_i \leq \frac{m}{2} - \sqrt{km}\right\}\right] &\leq \sum_{i=1}^{k} \mathbf{Pr}\left[X_i - \frac{m}{2} \leq -\sqrt{km}\right] \\ &\leq \sum_{i=1}^{k} \mathbf{Pr}\left[\left|X_i - \mathbb{E}\, X_i\right| \geq \sqrt{km}\right] \\ &\leq \sum_{i=1}^{k} \frac{\mathbf{Var}\, X_i}{km} = \frac{1}{4} < 1.\end{aligned}$$

Thus, there is a positive probability that the random bipartition works for all G_i, implying that there is such a bipartition. $\square$