These problems are from http://math.cmu.edu/~cocox/teaching/discrete20/extra.pdf

Problem 1. Let $G = (V, E)$ be a connected, weighted graph with weight function $w\colon E \to \mathbb{R}$. Let $\mathcal{T}$ denote the set of all spanning trees of G and let $\mathcal{T}_{\min} \subseteq \mathcal{T}$ be the set of all min-weight spanning trees.

Build a graph $\mathcal{G} = (\mathcal{T}, \mathcal{E})$ where $\{T, F\} \in \mathcal{E}$ if there are edges $e \in E(T) \setminus E(F)$ and $f \in E(F) \setminus E(T)$ such that $F = T - e + f$ (equivalently $T = F - f + e$). Furthermore, let $\mathcal{G}_{\min}$ be the subgraph of $\mathcal{G}$ induced by $\mathcal{T}_{\min}$ (i.e. $\mathcal{G} = \big(\mathcal{T}_{\min}, \mathcal{E} \cap \binom{\mathcal{T}_{\min}}{2}\big)$). Prove the following:

1. For any $T, F \in \mathcal{T}$, we have $\{T, F\} \in \mathcal{E}$ if and only if $|E(T) \triangle E(F)| = 2$.
2. For any $T \in \mathcal{T}$ and $T^* \in \mathcal{T}_{\min}$, there is a path $(T = T_1, \ldots, T_k = T^*)$ in $\mathcal{G}$ where $w(T_i) \geq w(T_{i+1})$ for all $i \in [k-1]$.
3. For any $T \in \mathcal{T} \setminus \mathcal{T}_{\min}$, there is $F \in \mathcal{T}$ with $\{T, F\} \in \mathcal{E}$ and $w(F) < w(T)$.
4. $\mathcal{G}$ is connected.
5. $\mathcal{G}_{\min}$ is connected.

Problem 2. Let $G = (V, E)$ be a weighted graph with weight function $w\colon E \to \mathbb{R}$. Recall that for a subgraph H, the weight of H is defined to be $w(H) \overset{\text{def}}{=} \sum_{e \in E(H)} w(e)$. We could additionally define the *multiplicative weight* of H to be $w^*(H) \overset{\text{def}}{=} \prod_{e \in E(H)} w(e)$.

Let G be a connected, weighted graph with weight function $w\colon E \to \mathbb{R}_{>0}$. Prove that T is a spanning tree which minimizes $w(T)$ if and only if T minimizes $w^*(T)$.

In other words, using the notation from Problem 1, if $\mathcal{T}^*_{\min}$ denotes the set of all min-multiplicative-weight spanning trees, then $\mathcal{T}^*_{\min} = \mathcal{T}_{\min}$, provided all edge-weights are positive.

Problem 3. For a graph G, let $\operatorname{conn}(G)$ denote the set of connected components of G.

Let $G = (V, E)$ be a weighted graph with weight function $w\colon E \to \mathbb{R}$ and consider the following algorithm:

procedure BORŮVKA(G)
  $F \leftarrow (V, \varnothing)$
  while $|\operatorname{conn}(F)| > 1$ **do**
    $F \leftarrow F + \sum_{C \in \operatorname{conn}(F)} e(C)$ ▷ $e(C)$ is the cheapest edge with exactly one vertex in C
  end while
  return F
end procedure

1. Prove that if G is connected and each edge has a distinct weight, then BORŮVKA(G) returns a min-weight spanning tree of G.
2. Why was it necessary to assume that each edge has a distinct weight?
3. How can we fix the algorithm to handle the case wherein G contains edges of equal weights?