MATHEMATICAL TRIPOS PART III, 2013
REPRESENTATION THEORY

This sheet deals mostly with polynomial invariant theory. Work over $\mathbb{C}$.

1 Let W be finite-dimensional over $\mathbb{C}$. Check that the coordinate functions $x_1, \ldots, x_m \in \mathbb{C}[W]$ are algebraically independent (equivalently, if $f(a_1, \ldots, a_m) = 0$ for a polynomial f and all $a = (a_1, \ldots, a_m) \in \mathbb{C}^m$ then f is the zero polynomial). Hint: use induction on the number of variables and the fact that a non-zero polynomial in one variable has only finitely many zeroes.

2 (a) Show that the natural representation of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}^2$ is self-dual (i.e. equivalent to its dual representation). Hint: find a non-singular matrix P such that $(A^{-1})^t = PAP^{-1}$.

(b) Show that this representation has two orbits.

3 Consider GL_2 acting on $\mathbb{C}[X, Y]$ (induced by the natural representation of GL_2 on $\mathbb{C}^2$).

(a) What is the image of X and Y under the action of any $g \in \mathrm{GL}_2$?

(b) Calculate $\mathbb{C}[X, Y]^{\mathrm{GL}_2}$ and $\mathbb{C}[X, Y]^{\mathrm{SL}_2}$.

(c) Calculate $\mathbb{C}[X, Y]^U$ where U is the subgroup of upper triangular unipotent matrices.

4 Consider the representation of $\mathrm{SL}_n(\mathbb{C})$ in its action on $\mathrm{M}_n(\mathbb{C})$, as discussed in lectures. Prove that the invariant ring is generated by the determinant. You may need the fact that GL_n is Zariski-dense in M_n.

5 Determine the invariant rings of $\mathbb{C}[\mathrm{M}_2(\mathbb{C})]^U$ and $\mathbb{C}[\mathrm{M}_2(\mathbb{C})]^T$ under left multiplication by the subgroup U of upper triangular unipotent matrices and the 2-torus $T = \mathrm{diag}(t, t^{-1})$ of diagonal matrices of SL_2.

6 Let T_n be the subgroup of GL_n of non-singular diagonal matrices. Choose the standard basis in $W = \mathbb{C}^n$ and the dual basis in W^* and thus identify the coordinate ring $\mathbb{C}[W \oplus W^*]$ with $\mathbb{C}[x_1, \ldots x_n, \zeta_1, \ldots, \zeta_n]$. Show that

$$\mathbb{C}[W \oplus W^*]^{T_n} = \mathbb{C}[x_1\zeta_1, \ldots, x_n\zeta_n].$$

What happens if you replace T_n by the subgroup T_n' of diagonal matrices with determinant 1?

7 Recall that for each $j \geqslant 1$ the symmetric functions $n_j(\mathbf{x}) = x_1^j + x_2^j + \cdots x_n^j$ are known as *power sums* (or *Newton functions*). As before the e_j are the elementary symmetric functions.

(a) Show that

$$(-1)^{j+1} j e_j = n_j - e_1 n_{j-1} + e_2 n_{j-2} - \cdots + (-1)^{j-1} e_{j-1} n_1$$

for all $j = 1 \ldots, n$. This is known as the Newton-Girard identity. [Hint: let $j = n$, consider $f(t) = \prod_i (t - x_i)$ and calculate $\sum_i f(x_i)$. For $j < n$ what can you say about the right-hand side?]

(b) Here is Weyl's original proof[1] of the identity in (a). Define the polynomial

$$\psi(t) = \prod_{i=1}^{n} (1 - t x_i) = 1 - e_1 t + e_2 t^2 - \cdots + (-1)^n e_n t^n,$$

where the e_i are the elementary symmetric functions. Determine its logarithmic derivative $-\frac{\psi'(t)}{\psi(t)}$ as a formal power series. Deduce (a).

(c) Show that (in characteristic 0) the power sums generate the symmetric functions.

8 (a) Let A be a commutative algebra and let G be a group algebra of automorphisms of A. Assume the representation of G on A is completely reducible. Show that the subalgebra A^G of invariants has a canonical G-stable complement and the corresponding G-equivariant projection $\pi : A \to A^G$ satisfies $\pi(hf) = h\pi(f)$ for $h \in A^G, f \in A$. This projection is sometimes called the *Reynolds operator*.

(b) Let $A = \oplus_{j \geqslant 0} A_j$ be a graded K-algbbra (meaning $A_i A_j \subset A_{i+j}$). Assume the idea $A^+ = \oplus_{j>0} A_j$ is finitely-generated. Show that A is finitely-generated as an algebra over A_0. (In fact if the ideal A^+ is generated by the homogeneous elements $a_1, \ldots, a_n$, then $A = A_0[a_1, \ldots, a_n]$.)

(c) Deduce Hilbert's theorem: if W is a G-module and the representation of G on $K[W]$ is completely reducible then the invariant ring $K[W]^G$ is finitely-generated. [Hint: you will need to use Hilbert's Basis Theorem.] This shows also that $K[W]^G$ is noetherian.

(d) Hilbert's Theorem is applicable to finite groups provided Maschke's Theorem holds (i.e we require that $|G|$ is invertible in K). In the special case of characteristic 0, show that for any representation W of G, the ring of invariants $K[W]^G$ is generated by invariants of degree less than or equal to $|G|$ (Noether's Theorem).

Schmidt introduced a numerical invariant $\beta(G)$ for every finite group G. It is defined to be the minimal number m such that for every representation W of G, the invariant ring $K[W]^G$ is generated by the invariants of degree $\leqslant m$. With this definition, Noether's Theorem asserts that $\beta(G) \leqslant |G|$. In fact Schmidt showed that $\beta(G) = |G|$ if and only if G is cyclic[2]. It is also a fact that if G is actually abelain then $\beta(G)$ coincides with the so-called Davenport constant.

In the next few questions we'll write $V = \mathbb{C}^2$, $G = \mathrm{SL}(V) = \mathrm{SL}_2(\mathbb{C})$, and

$$C_n = H_{\mathbb{C},n}(V, \mathbb{C}) \cong (S^n V)^* \cong S^n(V^*),$$

which can be identified with the set of homogeneous polynomials of degree n in variables X_1, X_2. Note that $\{C_n : n \in \mathbb{N}\}$ are the non-isomorphic simple $\mathbb{C}G$-modules. For a fixed n, recall that a *covariant* for C_n is a polynomial $\mathbb{C}\mathrm{SL}(V)$-invariant $C_n \oplus V \to \mathbb{C}$. The following questions sketch a classification of the generators for $\mathbb{C}[C_n \otimes V)]^G$.

[1]Weyl, The Classical Groups, II.A.3

[2]see *Finite groups and invariant theory* in Séminaire d'Algèbre Paul Dubreil et M.-P. Malliavin (Springer Lecture Notes 1478, Springer 1991)

9 Suppose f, g are functions of X_1, X_2 and $r \in \mathbb{N}$. Define the rth *transvectant* of f, g as

$$\tau_r(f,g) = \sum_{j=0}^{r} \frac{(-1)^j}{j!(r-j)!} \frac{\partial^r f}{\partial X_1^{r-j} \partial X_2^j} \frac{\partial^r g}{\partial X_1^j \partial X_2^{r-j}}.$$

$$= \frac{1}{r!} \left(\frac{\partial}{\partial X_1} \frac{\partial}{\partial Y_2} - \frac{\partial}{\partial X_2} \frac{\partial}{\partial Y_1} \right)^r (f(X_1, X_2) g(Y_1, Y_2) \,|_{Y_1 = X_1, Y_2 = X_2}.$$

You should observe $\tau_0(f,g) = fg$, $\tau_1(f,g)$ is the Jacobian of f, g and $\tau_2(f,f)$ is the Hessian of f. Show that

(a) if f, g are homogeneous polynomials of degrees p, q then $\tau_r(f,g) = 0$ unless $r \leqslant \min\{p,q\}$, in which case it is a homogeneous polynomial of degree $p + q - 2r$;

(b) if $r \leqslant \min\{p,q\}$ then the map $\tau_r : C_p \otimes C_q \to C_{p+q-2r}$ sending $f \otimes g \mapsto \tau_r(f,g)$ is a non-zero map of $\mathbb{C}G$-modules;

(c) any $\mathbb{C}G$-module map $C_p \otimes C_q \to C_{p+q-2r}$ is a multiple of τ_r. [Hint: use Clebsch-Gordan];

(d) the τ_r give an isomorphism of $\mathbb{C}G$-modules

$$C_p \otimes C_q \to \bigoplus_{r=0}^{\min\{p,q\}} C_{p+q-2r}.$$

10 Let $p, q, n, r \in \mathbb{N}$ and $r \leqslant \min\{q, n\}$. Set $N = \max\{0, r-p\}$. Show that there are scalars $\lambda_N, \ldots, \lambda_r \in \mathbb{C}$, with $\lambda_r \neq 0$, such that

$$f\tau_r(g,h) = \sum_{j=N}^{r} \lambda_j \tau_j(\tau_{r-j}(f,g), h),$$

for all $f \in C_p, g \in C_q, h \in C_n$. [HINT: Schur's Lemma.] Thus the τ_j are trying hard to be associative.

11 Fix $n \in \mathbb{N}$. Let $R = \mathbb{C}[C_n \oplus V]$, so that R^G is the set of covariants of C_n. If $\varphi \in R$ and $f \in C_n$, define $\varphi(f) \in \mathbb{C}[V]$ by $\varphi(f)(x) = \varphi(f,x)$. If $r \in \mathbb{N}$ and $\phi, \psi \in R$ define $\tau_r(\phi, \psi)$ as the map $C_n \oplus V \to \mathbb{C}$ by $(f,x) \mapsto \tau_r(\phi(f), \psi(f))(x)$. Finally if $d, i \in \mathbb{N}$, let R_{di} be the set of all $\phi \in R$ which are homogeneous, of degree d in C_n and degree i in V. Show that

(a) R_{di} are $\mathbb{C}G$-submodules of R, and R is graded, $R = \oplus_{d,i=0}^{\infty} R_{di}$, hence so too is R^G;

(b) $R_{di} R_{ej} \subseteq R_{d+e, i+j}$ and $\tau_r(R_{di}, R_{ej}) \subseteq R_{d+e, i+j-2r}$ for $r \leqslant \min\{i,j\}$;

(c) the assignment $\phi \mapsto (f \mapsto \phi(f))$ induces an isomorphism

$$R_{di}^G \cong \hom_{\mathbb{C}G,d}(C_n, C_i),$$

and deduce that

(d) any covariant $\theta \in R_{di}^G$ $(d \geqslant 1)$ can be expressed as a linear combination

$$\theta = \sum_{r=0}^{\min\{n,i\}} \tau_{n-r}(\phi_r, E)$$

with $\phi_r \in R_{d-1,n+i-2r}^G$ and E is the evaluation map $E(f,v) = f(v)$ [you should note that (c) says that $R_{1,i}^G = \mathbb{C}.E$ when $i = n$ and 0 if $i \neq n$.]

12 [Gordan's Theorem[3] - a weak form] If S is a $\mathbb{C}$-subalgebra of R^G with the property that $\tau_r(\phi, E) \in S$ whenever $r \in \mathbb{N}$ and $\phi \in S$, then $S = R^G$. [Remark: after the last question the proof should reduce to a two- line argument.]

13 [REALLY HARD] This uses Gordan's Theorem to find a set of generators of R^G in the case $n = 3$. Given $n = 3$, show that R^G is generated by the covariants

$E \in R^G_{13}$;
$H = \tau_2(E, E)$, the Hessian, $\in R^G_{22}$;
$t = \tau_1(H, E) \in R^G_{33}$;
$D = \tau_3(t, E)$, the discriminant ($\times 48$), $\in R^G_{40}$

14 [REALLY REALLY HARD[4]] Take $n = 4$ and repeat the last question to generate R^G using quartic forms.

Classically, the (still unsolved) problem of computing all covariants of binary forms of degree n was tackled by something called the symbolic method (using polarisation to reduce it to the FFT). Associated to the symbolic method is the symbolic notation, designed to make the calculations easier, but still non-trivial. You can consult any old text on Invariant Theory, such as Grace and Young if interested.

SM, Lent Term 2013
Comments on and corrections to this sheet may be emailed to `sm@dpmms.cam.ac.uk`

[3]Paul Gordan, dubbed the 'King of Invariant Theory' is perhaps better known for being Emmy Noether's thesis advisor. The result appeared in 1868 in Crelle's Journal as *Beweis, dass jede Covariante und Invariante einer binären Form eine ganze Funktion mit numerischen Coeffizienten einer endlichen Anzahl solcher Formen ist.*

[4]J.J. Sylvester's collected works (four volumes, edited by H.F. Baker) include tables with details about higher degree forms. However his published tables for forms of degree larger than 6 appear to be totally wrong. Corrections appear in von Gall (1888) and Dixmier/Lazard (1986), Shioda (1967) and Brouwer/Popoviciu (2010).