MATHEMATICAL TRIPOS PART III, 2013
REPRESENTATION THEORY

This sheet deals with the representation theory and combinatorics of S_n and of GL_n. Work over $\mathbb{C}$ unless told otherwise.

1 Let H be a subgroup of S_n; define the element $H^- = \sum_{h \in H} \varepsilon_h h$ which lies in the group algebra $\mathbb{C}S_n$. Prove that

$$\pi H^- = H^- \pi = \varepsilon_\pi H^- \quad \text{for any } \pi \in H,$$

and

$$\pi H^- \pi^{-1} = (\pi H \pi^{-1})^- \quad \text{for any } \pi \in S_n.$$

2 Let r be a natural number. Let $\lambda = (\lambda_1, \dots \lambda_\ell)$ be a sequence of positive integers. We call λ a *composition* of r into ℓ parts if $\sum_{i=1}^{\ell} \lambda_i = r$. The elements λ_i are called the *parts* of λ.

(a) Find a formula for the number of compositions of r.

(b) Find a formula for the number of compositions of r with at most k parts.

3 Let r, s be non-negative integers. A partition of the form $\lambda = (r, 1^s)$ is called a *hook* partition; we will write $H(n)$ for the set of all hook partitions of n. Let f^λ be the number of standard tableaux of shape λ.

(a) Find a formula for the number of standard tableaux of shape $\lambda = (r, 1^s)$ where λ is a partition of n. Justify your answer.

(b) Prove that $\sum_{\lambda \in H(n)} f^\lambda = 2^{n-1}$.

(c) Prove that $\sum_{\lambda \in H(n)} (f^\lambda)^2 = \binom{2(n-1)}{n-1}$.

4 Let $n \in \mathbb{N}$, and let $\lambda = (\lambda_i)_{i \in \mathbb{N}}$ and $\mu = (\mu_j)_{j \in \mathbb{N}}$ be partitions of n. The *conjugate partition* λ' is defined to be the partition $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ where $\lambda'_i = |\{j | \lambda_j \geqslant i\}|$. The *dominance order* on the set of partitions of n is defined as follows:

$\lambda \trianglelefteq \mu$ if and only if for all $i \in \mathbb{N}$:

$$\sum_{j \leqslant i} \lambda_j \leqslant \sum_{j \leqslant i} \mu_j.$$

(a) Prove that $\lambda \trianglerighteq \mu$ if and only if $\mu' \trianglerighteq \lambda'$.

(b) We say that a partition λ is *self-conjugate* if $\lambda = \lambda'$. Show that there are as many self-conjugate partitions of n as there are partitions of n with distinct odd parts (i.e. all λ_i are odd and distinct).

5 (a) Assuming $k \geqslant 3$, prove that then $f^{(3^k)} \geqslant 3k$.

(b) Let $\lambda = (l, \dots, l) = (l^k)$ be a partition of $n = kl$ and $\tilde{\lambda} = (l+1, \dots, l+1)$ be a partition of $n + k$. Prove (by using the Young-Frobenius formula (5.1)) that

$$\frac{f^{\tilde{\lambda}}}{f^\lambda} = \frac{(n+k)!}{n!} \cdot \frac{l!}{(l+k)!}.$$

(c) Let $k, l \geqslant 3$ with $kl = n$. Prove that then $f^{(l^k)} \geqslant n$.

6 Let C_k be the set of all standard tableaux of shape (k, k) whose entry in box (2,1) is some odd number.

(a) Find $|C_k|$.

(b) For a partition $\lambda = (r, s)$ with two parts, write down $f^{(r,s)}$ (e.g. from the Young-Frobenius formula). Can you find $\lim_{k\to\infty} |C_k|/f^{(k,k)}$?

7 In this question $\lambda = (\lambda_1 \lambda_2, \ldots, \lambda_k)$ is a partition of n, and M^λ is the permutation module with basis the λ-tabloids (a λ-tabloid is an equivalence class of numberings of a Young diagram (with distinct numbers 1,...,n), two being equivalent if the corresponding rows containing the same entries.)

(a) Show that M^λ is isomorphic to the permutation module $\mathbb{C}\Delta$ where $\Delta = \cos(S_n : S_\lambda)$, (the coset space with S_λ the Young subgroup).

(b) Show that

$$\dim M^\lambda = \frac{n!}{\prod_{i=1}^k \lambda_i!} \, .$$

(c) In the case $\lambda = (n - 2, 2)$ has two parts, show that $M^\lambda \cong \mathbb{C}\Omega^{(2)}$ where $\Omega^{(2)}$ is the set of 2-element subsets of $\Omega = \{1, \ldots, n\}$ (with the natural action).

8 For a λ-tableau T with n boxes, define the signed column sum $\kappa_T = \sum_{\sigma \in C_T} \varepsilon_\sigma \sigma$. Let $e_T = \kappa_T\{T\}$ be the λ-polytabloid. Let 1 denote the identity element of S_n.

(a) List the columns of T as $C_1, C_2, \ldots, C_s$. Prove that $\kappa_T = \kappa_{C_1} \cdots \kappa_{C_s}$.

(b) Let $H \leqslant S_n$ and let $(a\ b) \in H$. Define $H^- = \sum_{h\in H} \varepsilon_h h$, as in Q.1. Prove that $H^- = s \cdot (1 - (a\ b))$ for some $s \in \mathbb{C}S_n$.

(c) Let $\pi \in S_n$ and $\sigma \in C_T$. Show that

$$\kappa_{\pi T} = \pi \kappa_T \pi^{-1}, \quad \kappa_T \sigma = \sigma \kappa_T = \varepsilon_\sigma \kappa_T$$

and

$$e_{\pi T} = \pi e_T; \ ; \sigma e_T = \varepsilon_\sigma e_T.$$

9 *[The Murnaghan-Nakayama Rule.]

Part (a) below is an optional task for enthusiasts only. You will need to read about James' method (lecture notes pp 79–83) using the Littlewood-Richardson Rule. A good account appears in Sagan's Springer GTM, p.182. This beautiful result has generalisations to the irreducible characters of exotic structures called Iwahori-Hecke algebras of type A_{n-1} due to Ram (who appealed to Schur-Weyl duality), and it appears to have been known to Young in his work on hyperoctahedral groups (the so-called Weyl groups of type B_n).

(a) Consider the character $\chi^\lambda(w) =: \chi^\lambda_\mu$ where $w \in S_n$ has cycle type $(\mu_1, \mu_2, \ldots, \mu_\ell)$. Suppose there are precisely t partitions $\lambda(1), \ldots, \lambda(t)$ which are obtained by removing a rim hook of length μ_1 from $[\lambda]$. Find an expression for χ^λ_μ in terms of $\chi^{\lambda(i)}_{(\mu_2,\mu_3\ldots,\mu_\ell)}$.

(b) Let $n \geqslant 6$. Prove that

$$\chi^{(n-3,3)}_{(2,1^{n-2})} = (n - 3)(n - 4)(n - 5)/6.$$

10 Given some $\mathbb{C}S_n$-module M, with action written as $\pi \cdot m$ ($\pi \in S_n, m \in M$). Define a new $\mathbb{C}S_n$-module M^ε as the module whose underlying vector space is M and with new action $\pi * m := \varepsilon_\pi \pi \cdot m$ ($\pi \in S_n, m \in M$).

(a) Verify that M^ε is a $\mathbb{C}S_n$-module.

(b) Prove that M is irreducible if and only if M^ε is irreducible.

(c) Let χ be the character of M, and let $\chi^{(1^n)}$ be the character of the Specht module $S^{(1^n)}$. Prove that the character χ^ε of M^ε is given by $\chi^\varepsilon(\pi) = \chi^{(1^n)}(\pi)$ for $\pi \in S_n$.
[Why do you now know that the trivial module $S^{(n)}$ and the sign module $S^{(1^n)}$ are the only 1-dimensional $\mathbb{C}S_n$-modules.]

11 Let ρ :GL(V) $\to$GL$_n(\mathbb{C})$ be an irreducible polynomial representation. Show that the matrix entries $\rho_{ij} \in \mathbb{C}[\mathrm{GL}(V)]$ are homogeneous polynomials of the same degree. More generall, show that every polynomial representation ρ is a direct sum of representations $\rho^{(i)}$ whose matrix entries are homogeneous polynomials of degree i.

12 Give a proof of the result proved in lectures that affine n-space $\mathbb{A}^n$ is irreducible <u>without</u> appealing to the Nullstellensatz.

13 Prove that the 1-dimensional rational representations of $\mathbb{C}^*$ =GL$_1(\mathbb{C})$ are power maps of the form $u \mapsto u^r$ where $r \in \mathbb{Z}$.

14 (a) Show that the set of diagonalisable matrices is Zariski dense in M$_n(\mathbb{C})$ (hence every invariant function on M$_n(\mathbb{C})$ is completely determined by its restriction to T_n, the diagonal matrices. [Actually this is nothing more than Jordan decomposition; if you work a bit harder you can show this is true over any field, not necessarily algebraically closed].

(b) Let ρ :GL$_n$ $\to$GL(W) be a rational representation. Show that ρ is polynomial if and only if $\rho|_{T_n}$ is polynomial. Show also that ρ is polynomial if and only if its character χ_ρ is a polynomial.

(c) Deduce that every 1-dimensional rational representation of GL(V) is of the form $\det^r$:GL(V) $\to$GL$_1$, $r \in \mathbb{Z}$.

15 Consider the linear action of GL(V) on End(V) by conjugation.

(a) What are the orbits of this action? Any element $\alpha \in$End(V) has a characteristic polynomial of the form $p_\alpha(t) = t^n + \sum_{i=1}^n (-1)^i e_i(\alpha) t^{n-i}$, where $n = \dim V$, showing that the e_i are invariant polynomial functions on End(V).

(b) [HARD] Find a proof of the fact that the ring of invariants for the conjugation action is generated by the s_i without using properties of symmetric functions.

(c) Show that the functions Tr$_1 \ldots$,Tr$_n$ generate the invariant ring $\mathbb{C}[\mathrm{End}(V)]^{GL(V)}$. Here Tr$_k$:End$(V) \to \mathbb{C}$ is the map $A \mapsto$TrA^k, $(k = 1, 2, \ldots)$.

SM, Lent Term 2013
Comments on and corrections to this sheet may be emailed to `sm@dpmms.cam.ac.uk`