Cubical sets

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Contents

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1 The cube category

I will give three definitions of the cube category in decreasing order of concreteness. The first is a definition as a category of 'concrete' cubes and maps between them; the second is a similar definition, which is more commonplace; and the third is a snappy category theoretic definition. The first definition lets us really get a picture of what is happening, and makes the definition of *geometric realisation* easy to understand. The third definition is completely abstract and drives the point home that what we are doing is in some way 'correct'.

First definition: cubes

The first definition appears in [Ant02]. Let I denote the interval $[0,1]$. For $n \geq 0$, the *n-cube* I^n is simply the product

$$I^n = \underbrace{I \times I \times \cdots \times I}_{n \text{ times}} \subseteq \mathbb{R}^n$$

For example, I^0 is a point, I^1 is a line, I^2 is a (filled) square, I^3 is a cube, and so on. The topological structure of n-cubes plays no part in the definition of cubical sets, but it will facilitate the definition of the so-called *geometric realisation* later on (Definition 2.4).

Definition 1.1. A *face map* is a map $\delta_i^\varepsilon(n) : I^n \to I^{n+1}$, defined by

$$\delta_i^\varepsilon(n)(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, \varepsilon, x_i, \ldots, x_n)$$

where $\varepsilon \in \{0,1\}$ and $1 \leq i \leq n$. When n is clear from context we will suppress it.

Intuitively, a face map maps the n-cube I^n to some n-dimensional face $\delta_i^\varepsilon(I^n)$ of the $(n+1)$-cube. The direction that $\delta_i^\varepsilon(I^n)$ points is determined by i, and the position is determined by ε.

Definition 1.2. A *degeneracy map* is a map $e_i(n) : I^n \to I^{n-1}$, defined by

$$e_i(n)(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

where $1 \leq i \leq n$. When n is clear from context we will suppress it.

Intuitively, a degeneracy map flattens the cube along a dimension specified by i.

Proposition 1.3. Given $n \geq 0$ and $i < j$, the following diagrams commute

$$
\begin{array}{ccc}
I^n & \xrightarrow{\delta_i^\varepsilon} & I^{n+1} \\
{\scriptstyle \delta_{j-1}^\omega}\downarrow & & \downarrow{\scriptstyle \delta_j^\omega} \\
I^{n+1} & \xrightarrow[\delta_i^\varepsilon]{} & I^{n+2}
\end{array}
\qquad\qquad
\begin{array}{ccc}
I^{n+2} & \xrightarrow{e_j} & I^{n+1} \\
{\scriptstyle e_i}\downarrow & & \downarrow{\scriptstyle e_i} \\
I^{n+1} & \xrightarrow[e_{j-1}]{} & I^n
\end{array}
$$

Moreover the face and degeneracy maps interact in the following ways:

$$
\begin{array}{ccc}
I^n & \xrightarrow{e_{j-1}} & I^{n-1} \\
{\scriptstyle \delta_i^\varepsilon}\downarrow & & \downarrow{\scriptstyle \delta_i^\varepsilon} \\
I^{n+1} & \xrightarrow[e_j]{} & I^n
\end{array}
\qquad
\begin{array}{ccc}
I^n & & \\
{\scriptstyle \delta_i^\varepsilon}\downarrow & \searrow{\scriptstyle \mathsf{id}} & \\
I^{n+1} & \xrightarrow[e_j]{} & I^n
\end{array}
\qquad
\begin{array}{ccc}
I^n & \xrightarrow{e_j} & I^{n-1} \\
{\scriptstyle \delta_i^\varepsilon}\downarrow & & \downarrow{\scriptstyle \delta_{i-1}^\varepsilon} \\
I^{n+1} & \xrightarrow[e_j]{} & I^n
\end{array}
$$

$(i < j)$ $\qquad$ $(i = j)$ $\qquad$ $(i > j)$

In particular, the cubes equipped with these maps forms a category.

Definition 1.4. The *cube category* $\square$ is the subcategory of Top defined by

- $\mathrm{ob}(\square) = \{I^n : n < \omega\}$
- $\mathrm{mor}(\square)$ is generated by the face and degeneracy maps

A corollary of Proposition 1.3 is that any morphism $\lambda : I^n \to I^m$ in $\square$ can be written uniquely as a composite of face and degeneracy maps

$$\lambda = \delta_{i_k}^{\varepsilon_k} \circ \cdots \circ \delta_{i_1}^{\varepsilon_1} \circ e_{j_1} \circ \cdots \circ e_{j_\ell} : I^n \to \cdots \to I^{n-\ell} \to \cdots \to I^{n-\ell+k} = I^m$$

where $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_\ell$, and $\varepsilon_i \in \{0, 1\}$ for each i.

Second definition: sequences

An alternative but very similar definition is as follows. Now an n-cube is the set $2^n = \{0, 1\}^n$ of sequences of 0s and 1s of length n. Face and degeneracy maps are defined as in the first definition, and satisfy the same properties as in 1.3.

Definition 1.5. The *cube category* $\square$ is the subcategory of Set defined by

- $\mathrm{ob}(\square) = \{2^n : n < \omega\}$
- $\mathrm{mor}(\square)$ is generated by the face and degeneracy maps

Since this definition is so similar to Definition 1.4, I won't dwell on it much further, but I will point out some niceties:

- The face map δ_i^ε can now be described as 'inserting ε into the i^{th} coordinate'.
- The degeneracy map e_i can now be described as 'erasing the i^{th} coordinate'.
- In this formulation, everything is a concrete finite object, so is more amenable to computation than Definition 1.4.

Third definition: abstract nonsense

This is the definition given on nLab.

Definition 1.6. The *cube category* is the initial strict monoidal category $(\square, \otimes, 1)$ equipped with

- an object int ('interval')
- morphisms $\iota_0, \iota_1 : \mathbf{1} \to \mathsf{int}$ ('inclusions of end-points')
- a map $p : \mathsf{int} \to \mathbf{1}$ ('projection')

satisfying $p \circ \iota_0 = p \circ \iota_1 = \mathsf{id}_{\mathbf{1}}$.

The definition has the virtue of being abstract: we don't rely on the category of sets (or topological spaces) in order to construct $\square$ this way. We have an abstract interval ('1-cube') with abstract endpoints. We obtain the 'n-cube' as $\mathsf{int}^{\otimes n}$. Then we obtain face and degeneracy maps as

$$\delta_i^\varepsilon = (\mathsf{id}^{\otimes(i-1)} \otimes \iota_\varepsilon \otimes \mathsf{id}^{\otimes(n+1-i)}) \circ u_i(n)^{-1}, \quad e_i = u_i(n-1) \circ (\mathsf{id}^{\otimes(i-1)} \otimes p \otimes \mathsf{id}^{\otimes(n-i)})$$

where $u_i(n) : \mathsf{int}^{\otimes(i-1)} \otimes \mathbf{1} \otimes \mathsf{int}^{\otimes(n+1-i)} \cong \mathsf{int}^{\otimes n}$ is the obvious isomorphism.

Theorem 1.7. Definitions 1.4, 1.5 and 1.6 are equivalent.

Proof. The equivalence of Definitions 1.4 and 1.5 is clear.

(1.4)$\to$(1.6). Suppose $\square$ is defined as in Definition 1.4. We can define a monoidal structure on $\square$ by setting $I^n \otimes I^m = I^{n+m}$ and $\mathbf{1} = I^0$. This defines an initial strict monoidal category. Define $\mathsf{int} = I^1$. The maps

$$\iota_0 = \delta_0^0 : I^0 \to I^1, \quad \iota_1 = \delta_0^1 : I^0 \to I^1, \quad p = e_1 : I^1 \to I^0$$

clearly satisfy the requirements of Definition 1.4.

(1.6)$\to$(1.4). Suppose $\square$ is defined as in Definition 1.6. Defining $\square'$ as in 1.4 yields the structure of an initial strict monoidal category as above, and hence $\square' \cong \square$. It is easy to check that the new face and degeneracy maps, as defined above, are transferred across the isomorphism appropriately. □

2 Cubical sets and their basic properties

Fix your favourite definition of $\square$ from Section 1. For the sake of notation, we'll write $[n]$ for the n-cube in $\square$, and we'll write $\delta_i^\varepsilon(n) : [n] \to [n+1]$ and $e_i(n) : [n] \to [n-1]$ for the face and degeneracy maps, respectively.

2.1 Definition and examples

Definition 2.1. A *cubical set* is a presheaf on $\square$, i.e. a functor $X : \square^{\mathrm{op}} \to \mathsf{Set}$. The category of *cubical sets*, denoted cSet, has cubical sets as objects and natural transformations between them as morphisms.

For a cubical set X, write $X_n = X([n])$. Given $\lambda : [n] \to [m]$ we'll write λ_X^* to denote $X(\lambda)$. When X is clear from context, we'll just write λ^*.

Let's examine this definition more closely and see what these objects look like.

- We can think of X as being a space, characterised by how cubes are mapped into it.
- We can think of each $\sigma \in X_n$ as being an n-cube in X.
- Each face map $\delta_i^\varepsilon : [n] \to [n+1]$ induces a function

  $$\partial_i^\varepsilon := (\delta_i^\varepsilon)^* : X_{n+1} \to X_n$$

  which takes an n-cube and collapses it onto its $(n-1)$-face determined by i and ε, forgetting the rest of the structure.

- Each degeneracy map $e_i : [n] \to [n-1]$ induces a function

$$\mu_i := e_i^* : X_{n-1} \to X_n$$

which takes an $(n-1)$-cube x to the n-cube built from two copies of x, glued together with identities along dimension i.

- A morphism $\theta : X \to Y$ in cSet is a collection of functions $\theta_n : X_n \to Y_n$ making the following square commute for all $\square$-morphisms $\lambda : [n] \to [m]$

$$\begin{array}{ccc} X_m & \xrightarrow{\lambda_X^*} & X_n \\ {\scriptstyle \theta_m}\big\downarrow & & \big\downarrow{\scriptstyle \theta_n} \\ Y_m & \xrightarrow[\lambda_Y^*]{} & Y_n \end{array}$$

In other words, θ is a collection of functions on cubes which preserve dimension, and respect the face and degeneracy maps.

What follows are some important examples of cubical sets.

Example 2.2. The *standard n-cube* $\square^n$ is defined to be the representable functor

$$\square^n = y[n] = \mathrm{Hom}_{\square}(-, [n]) : \square^{\mathrm{op}} \to \mathsf{Set}$$

where y is the Yoneda embedding $\square \to \mathsf{cSet}$.

Thus $\square^n_m = \{\text{morphisms } [m] \to [n]\}$ and $\lambda^*(f) = f \circ \lambda$.

Example 2.3. Let A be a topological space. The *singular cubical set* $S(A)$ of A is defined by

$$S(A)_n = \{\text{continuous maps } I^n \to A\}$$

and, given $\lambda : I^n \to I^m$ and $\sigma : I^m \to A$, we define

$$\lambda^*(\sigma) = \sigma \circ \lambda : I^n \to A$$

It is easy to verify that $S(A)$ defines a cubical set: if we have $I^n \xrightarrow{\lambda_1} I^m \xrightarrow{\lambda_2} I^k$ and $\sigma : I^m \to A$ then

$$(\lambda_2 \circ \lambda_1)^*(\sigma) = \sigma \circ (\lambda_2 \circ \lambda_1) = (\sigma \circ \lambda_2) \circ \lambda_1 = (\lambda_1^* \circ \lambda_2^*)(\sigma)$$

and clearly $(\mathsf{id}_{I^n})^* = \mathsf{id}_{S(A)_n}$, so $S(A)$ is a contravariant functor.

Note that here we *do* use the topological structure on I^n.

2.2 Geometric realisation

As mentioned before, we can think of a cubical set as a space. This idea is made concrete by introducing the idea of the geometric realisation of a cubical set.

Definition 2.4. Let X be a cubical set. The *geometric realisation* of X is the topological space $|X|$ defined by

$$|X| = \left(\coprod_{n<\omega} X_n \times I^n \right) \Big/ \sim$$

where $\sim$ is the equivalence relation defined by

$$\underbrace{(\lambda^*(\sigma), t)}_{\in X_n \times I^n} \sim \underbrace{(\sigma, \lambda(t))}_{\in X_m \times I^m} \text{ for all } m, n < \omega,\ \sigma \in X_m,\ t \in I^n,\ \lambda : I^n \to I^m$$

and each X_n is equipped with the discrete topology.

Now we really *are* mapping cubes into spaces! The space $|X|$ is built from lots of cubes, one n-cube $\{\sigma\} \times I^n$ for each $\sigma \in X_n$, with the equivalence relation $\sim$ taking care of face relations.

Proposition 2.5. For each $n \in \mathbb{N}$, $|\square^n| \cong I^n$.

Proof. For all $\sigma : I^k \to I^n$ and all $t \in I^k$ we have

$$(\sigma, t) = (\sigma^*(\mathsf{id}_{I^n}), t) \sim (\mathsf{id}_{I^n}, \sigma(t))$$

So the map $(\sigma, t) \mapsto \sigma(t)$ defines a homeomorphism $|\square^n| \cong I^n$. □

Theorem 2.6. Example 2.3 and Definition 2.4 give rise to functors

$$S : \mathsf{Top} \to \mathsf{cSet}, \quad |-| : \mathsf{cSet} \to \mathsf{Top}$$

Moreover, there is an adjunction $(|-| \dashv S)$.

Proof. S **defines a functor.** Given $f : A \to B$ in Top define $S(f) : S(A) \to S(B)$ by

$$S(f)_n(\sigma) = f \circ \sigma$$

Given $\lambda : I^n \to I^m$ and $\sigma : I^m \to X$ we then have

$$S(f)_n(\lambda_A^*(\sigma)) = S(f)_n(\sigma \circ \lambda) = f \circ (\sigma \circ \lambda) = (f \circ \sigma) \circ \lambda = \lambda_B^*(f \circ \sigma) = \lambda_B^*(S(f)_m(\sigma))$$

$$\begin{array}{ccc} S(A)_m & \xrightarrow{\lambda_A^*} & S(A)_n \\ {\scriptstyle S(f)_m}\downarrow & & \downarrow{\scriptstyle S(f)_n} \\ S(B)_m & \xrightarrow[\lambda_B^*]{} & S(B)_n \end{array}$$

so $S(f)$ is a natural transformation. It can be easily verified that $S(f \circ f') = S(f) \circ S(f')$ by pasting together the appropriate squares.

$|-|$ **defines a functor.** Given $\theta : X \to Y$ in cSet and $(\sigma, t) \in X_n \times I^n$ define

$$|\theta|(\sigma, t) = (\theta_n(\sigma), t)$$

This is well-defined, since

$$\theta(\sigma, \lambda(t)) = (\theta_n(\sigma), \lambda(t)) = (\lambda^*(\theta_n(\sigma)), t) = (\theta_m(\lambda^*(\sigma)), t) = \theta(\lambda^*(\sigma), t)$$

so θ respects $\sim$. It is clear that $|\theta \circ \theta'| = |\theta| \circ |\theta'|$ and $|1_X| = 1_{|X|}$ using the properties of natural transformations, e.g. $(\theta \circ \theta')_n = \theta_n \circ \theta'_n$.

Unit of the adjunction. For a cubical set X define $\eta_X : X \to S|X|$ as follows: for each n and each $\sigma \in X_n$, define

$$(\eta_X)_n(\sigma) = \widehat{\sigma}, \quad \text{where } \widehat{\sigma}(t) = (\sigma, t) \text{ for } t \in I^n$$

Note that $(\eta_X)_n$ defines a natural transformation: if $\lambda : I^n \to I^m$ in $\square$, $\sigma \in X_m$ and $t \in I^n$ then

$$(\eta_Y)_n(\lambda^*(\sigma))(t) = \widehat{\lambda^*(\sigma)}(t) = (\lambda^*(\sigma), t) = (\sigma, \lambda(t)) = \widehat{\sigma}(\lambda(t)) = \lambda^*(\widehat{\sigma})(t) = \lambda^*((\eta_X)_m(\sigma))(t)$$

so η_X is a morphism in cSet.

For η to define a natural transformation $1_{\mathsf{cSet}} \to S|-|$ we need to following diagram to commute for all $\theta : X \to Y$ in cSet:

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & S|X| \\ {\scriptstyle \theta}\downarrow & & \downarrow{\scriptstyle S|\theta|} \\ Y & \xrightarrow[\eta_Y]{} & S|Y| \end{array}$$

Fix $n \in \mathbb{N}$, $\sigma \in X_n$ and $t \in I^n$. Then

$$S|\theta|_n((\eta_X)_n(\sigma))(t) = S|\theta|_n(\widehat{\sigma})(t) = (|\theta| \circ \widehat{\sigma})(t) = |\theta|(\sigma, t)$$

$$= (\theta_n(\sigma), t) = \widehat{\theta_n(\sigma)}(t) = (\eta_Y)_n(\theta_n(\sigma))(t)$$

so $S|\theta| \circ \eta_X = \eta_Y \circ \theta$ and η is natural.

Counit of the adjunction. For a topological space A define $\varepsilon_A : |S(A)| \to A$ by

$$\varepsilon_A(\sigma, t) = \sigma(t)$$

Then ε_A is well-defined since

$$\varepsilon_A(\lambda^*(\sigma), t) = \lambda^*(\sigma)(t) = \sigma(\lambda(t)) = \varepsilon_A(\sigma, \lambda(t))$$

For ε to define a natural transformation $|S(-)| \to 1_{\mathsf{Top}}$ we need to following diagram to commute for all $f : A \to B$ in Top:

$$\begin{array}{ccc} |S(A)| & \xrightarrow{\varepsilon_A} & A \\ {\scriptstyle |S(f)|}\downarrow & & \downarrow{\scriptstyle f} \\ |S(B)| & \xrightarrow[\varepsilon_B]{} & B \end{array}$$

Fix $\sigma : I^n \to A$ and $t \in I^n$. Then

$$f(\varepsilon_A(\sigma, t)) = f(\sigma(t)) = (f \circ \sigma)(t) = \varepsilon_B(f \circ \sigma, t) = \varepsilon_B(|S(f)|_n(\sigma), t) = \varepsilon_B(|S(f)|(\sigma, t))$$

so $f \circ \varepsilon_A = \varepsilon_B \circ |S(f)|$ and ε is a natural transformation.

Triangle identities. To complete the proof we need to prove that the triangle identities hold:

$$\begin{array}{ccc} SA & \xrightarrow{\eta_{SA}} & S|SA| \\ & {\scriptstyle \mathsf{id}}\searrow & \downarrow{\scriptstyle S\varepsilon_A} \\ & & SA \end{array} \qquad\qquad \begin{array}{ccc} |X| & \xrightarrow{|\eta_X|} & |S|X|| \\ & {\scriptstyle \mathsf{id}}\searrow & \downarrow{\scriptstyle \varepsilon_{|X|}} \\ & & |X| \end{array}$$

For the first, let $\sigma : I^n \to A$ and $t \in I^n$; then

$$(S\varepsilon_A)_n((\eta_{SA})_n(\sigma))(t) = (S\varepsilon_A)_n(\widehat{\sigma})(t) = (\varepsilon_A \circ \widehat{\sigma})(t) = \varepsilon_A(\sigma, t) = \sigma(t)$$

so $S\varepsilon_A \circ \eta_{SA} = \mathsf{id}_{SA}$.

For the second, let $(\sigma, t) \in X_n \times I^n$. Then

$$\varepsilon_{|X|}(|\eta_X|(\sigma,t)) = \varepsilon_{|X|}((\eta_X)_n(\sigma),t) = \varepsilon_{|X|}(\widehat{\sigma},t) = \widehat{\sigma}(t) = (\sigma,t)$$

so $\varepsilon_{|X|} \circ |\eta_X| = \mathsf{id}_{|X|}$.

Thus there is an adjunction $(|-| \dashv S)$. □

TL;DR. A summary of details given in the proof is as follows:

- The functor $|-| : \mathsf{cSet} \to \mathsf{Top}$ is defined by
  - $|X| = \left(\coprod_{n\in\mathbb{N}} X_n \times I^n\right) / \left((\lambda^*(\sigma), t) \sim (\sigma, \lambda(t))\right)$
  - $|\theta|(\sigma, t) = (\theta_n(\sigma), t)$
- The functor $S : \mathsf{Top} \to \mathsf{cSet}$ is defined by
  - $S(A)_n = \{\text{continuous functions } I^n \to A\}$
  - $S(f)_n(\sigma) = f \circ \sigma$ for $\sigma : I^n \to A$
- The unit $\eta : 1_{\mathsf{cSet}} \to S|-|$ is defined by

$$(\eta_X)_n(\sigma) = \widehat{\sigma}, \quad \text{where } \widehat{\sigma}(t) = (\sigma, t) \text{ for } \sigma \in X_n,\ t \in I^n$$

- The counit $\varepsilon : |S(-)| \to 1_{\mathsf{Top}}$ is defined by

$$\varepsilon_A(\sigma, t) = \sigma(t)$$

Given a cubical set X and a space A, this adjunction gives us a natural correspondence between maps $X \to SA$ and $|X| \to A$ as follows:

- $\theta : X \to SA$ corresponds with $\bar{\theta} : |X| \to SA$, where for $\sigma \in X_n$ and $t \in I^n$ we define

$$\bar{\theta}(\sigma, t) = \theta_n(\sigma)(t)$$

- $f : |X| \to A$ corresponds with $\bar{f} : X \to SA$, where for $n \in \mathbb{N}$ we define

$$\bar{f}_n(\sigma)(t) = f(\sigma, t)$$

Thus the adjunction gives us a kind of currying and uncurrying operation.

There is an abstract definition of geometric realisation, which doesn't rely on encoding any topological structure into the cube category $\square$. First we need to define a new functor.

Definition 2.7. Given a cubical set X, define the functor $X \cdot I : \square^{\mathrm{op}} \times \square \to \mathsf{Top}$ to be the composite

$$\square^{\mathrm{op}} \times \square \xrightarrow{X \times I} \mathsf{Set} \times \mathsf{Top} \xrightarrow{\mathsf{disc} \times \mathsf{id}} \mathsf{Top} \times \mathsf{Top} \xrightarrow{\mathsf{prod}} \mathsf{Top}$$

where

- $I : \square \to \mathsf{Top}$ is the functor taking the n-cube in $\square$ to I^n
- $\mathsf{disc} : \mathsf{Set} \to \mathsf{Top}$ is the functor that equips a set with the discrete topology
- $\mathsf{prod} : \mathsf{Top} \times \mathsf{Top} \to \mathsf{Top}$ is the functor that takes two spaces to their product, with the product topology

In particular, if $[m], [n] \in \square$ then $(X \cdot I)([m], [n]) = X_m \times I^n$.

Notice that if we use Definition 1.4 of $\square$ then I is just the inclusion functor $\square \hookrightarrow \mathsf{Top}$.

Definition 2.8. Let X be a cubical set. The *geometric realisation* of X is the coend

$$|X| = \int^{\square} X \cdot I$$

Spelling this out (see Appendix A.1): $|X|$ is a topological space equipped with, for each n, a map $\iota_n : X_n \times I^n \to |X|$ such that, for all $\lambda : I^n \to I^m$ in $\square$, the following diagram is a pushout:

$$\begin{array}{ccc} X_m \times I^n & \xrightarrow{\lambda^* \times \mathsf{id}} & X_n \times I^n \\ {\scriptstyle \mathsf{id} \times \lambda} \downarrow & & \downarrow {\scriptstyle \iota_n} \\ X_m \times I^m & \xrightarrow[\iota_m]{} & |X| \end{array}$$

This is *precisely* the assertion that $|X|$ is the coproduct of the $X_n \times I^n$, identifying $(\lambda^*(\sigma), t)$ and $(\sigma, \lambda(t))$ for all $\sigma \in X_m$ and $t \in I^n$.

2.3 Kan condition

Definition 2.9. A *cubical subset* of a cubical set X is a cubical set Y such that $Y_n \subseteq X_n$ for each n, inheriting the same face and degeneracy maps. That is, given a face map $\delta^\varepsilon : I^n \to I^{n+1}$ we have $\partial_i^\varepsilon[Y_{n+1}] \subseteq Y_n$, and given a degeneracy map $e_i : I^n \to I^{n-1}$ we have $\mu_i[Y_{n-1}] \subseteq Y_n$.

If Y is a cubical subset of X, we write $Y \subseteq X$. Note that $\subseteq$ defines a partial order on the set of cubical subsets of X.

Definition 2.10. Let X be a cubical set and $A \subseteq \bigcup_{n<\omega} X_n$ be a set of cubes in X. The cubical subset of X *generated by* A, denoted $\langle A \rangle$, is the $\subseteq$-least cubical subset of X containing A.

The purpose of these definitions is to translate the notion of *horn* from simplicial sets into the language of cubical sets.

Definition 2.11. Fix n and let $1 \leq i \leq n$ and $\varepsilon \in \{0, 1\}$. The (i, ε)-*box* of $\square^n$, denoted $\sqcap^n_{i,\varepsilon}$, is defined by

$$\sqcap^n_{i,\varepsilon} = \langle \{\delta^\omega_j : I^{n-1} \to I^n \mid (j, \omega) \neq (i, \varepsilon)\} \rangle$$

That is, $\sqcap^n_{i,\varepsilon}$ is the cubical subset of $\square^n$ generated by all the face maps except for δ^ε_i. This intuition is made concrete by taking the geometric realisation:

Proposition 2.12. The geometric realisation $|\sqcap^n_{i,\varepsilon}|$ is the union of the $(n-1)$-faces of the standard n-cube, save for the face determined by (i, ε). □

Definition 2.13. A cubical set X is *Kan* (or *satisfies the Kan condition*) if every map $f : \sqcap^n_{i,\varepsilon} \to X$ extends to a map $\widehat{f} : \square^n \to X$.

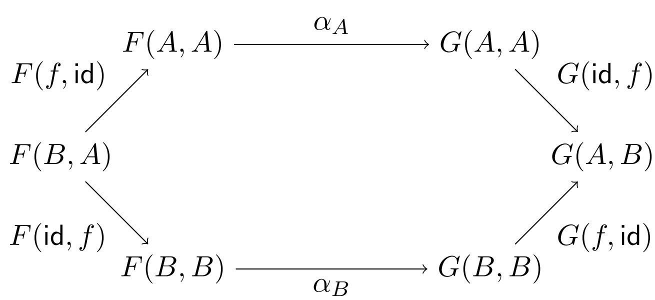
Appendices

A Category theory

A.1 Dinatural transformations, ends and coends

Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F, G : \mathcal{C}^{\mathrm{op}} \to \mathcal{C} \to \mathcal{D}$ be functors.

Definition A.1. A *dinatural transformation* $\alpha : F \to G$ is a collection of $\mathcal{D}$-morphisms $\alpha_A : F(A, A) \to G(A, A)$ for $\mathcal{D}$-objects A, such that for each $f : A \to B$ in $\mathcal{C}$ the following diagram commutes:

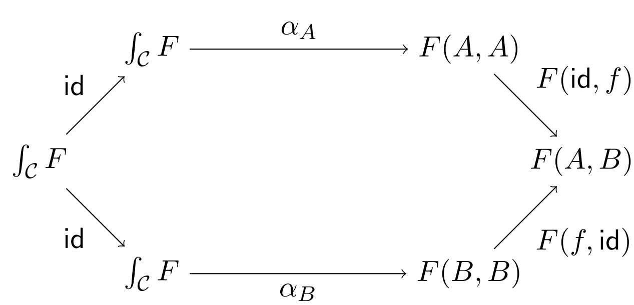


Every natural transformation $\eta : F \to G$ gives rise to a dinatural transformation $\alpha : F \to G$ via $\alpha_A = \eta_{(A,A)}$.

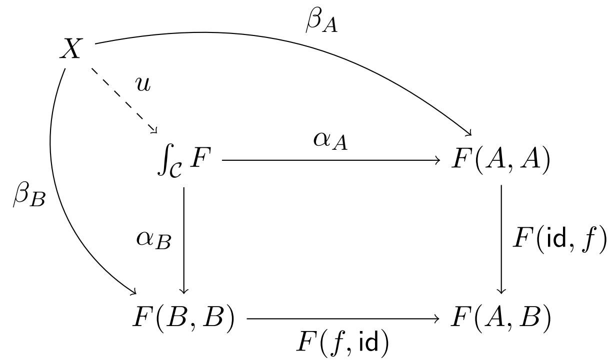
In what follows, given categories $\mathcal{C}, \mathcal{D}$ and a $\mathcal{D}$-object B, $[B]$ will the constant functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ with value B on morphisms and id_B on morphisms.

Definition A.2. Let $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ be a functor. The *end* of F is a $\mathcal{D}$-object $\int_{\mathcal{C}} F$ together with a universal dinatural transformation $\alpha : \left[\int_{\mathcal{C}} F\right] \to F$, i.e. such that if $\beta : [X] \to F$ is another dinatural transformation, with X a $\mathcal{D}$-object, then there is a unique $\mathcal{D}$-morphism $u : X \to \int_{\mathcal{C}} F$ such that $\beta_A = \alpha_A \circ u$ for all $\mathcal{C}$-objects A.

The corresponding dinatural transformation diagram, for $f : A \to B$ in $\mathcal{C}$, is



By collapsing the identity maps, the universal property is thus the assertion that the following diagram commutes, with u unique:



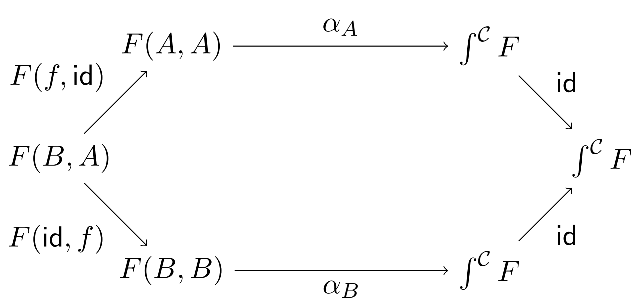
For instance, when $\mathcal{D} = \mathsf{Set}$, we have

$$\int_{\mathcal{C}} F = \{a : \mathrm{ob}(\mathcal{C}) \to \mathsf{Set} \mid F(\mathsf{id}, f)(a(A)) = F(f, \mathsf{id})(a(B)) \text{ for all } f : A \to B\}$$
$$\subseteq \prod_{A \in \mathrm{ob}(\mathcal{C})} F(A, A)$$

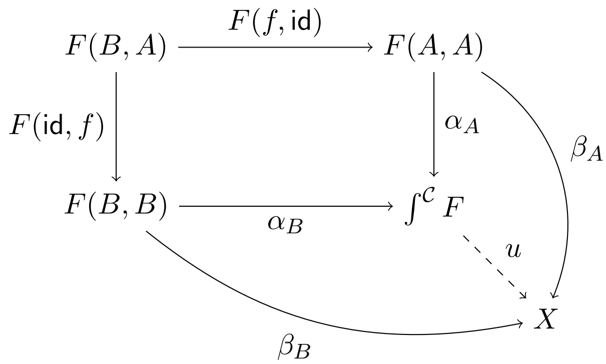
and α_A is the projection $\int_{\mathcal{C}} F \to F(A, A)$.

Definition A.3. Let $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ be a functor. The *coend* of F is a $\mathcal{D}$-object $\int^{\mathcal{C}} F$ together with a universal dinatural transformation $\alpha : F \to \left[\int^{\mathcal{C}} F\right]$, i.e. such that if $\beta : F \to [X]$ is another dinatural transformation, with X a $\mathcal{D}$-object, then there is a unique $\mathcal{D}$-morphism $u : \int^{\mathcal{C}} F \to X$ such that $\beta_A = u \circ \alpha_A$ for all $\mathcal{C}$-objects A.

The corresponding dinatural transformation diagram, for $f : A \to B$ in $\mathcal{C}$, is



By collapsing the identity maps, the universal property is thus the assertion that the following diagram commutes, with u unique:



For instance, when $\mathcal{D} = \mathsf{Set}$ (or Top), we have

$$\int^{\mathcal{C}} F = \coprod_{A \in \mathrm{ob}(\mathcal{C})} F(A,A) \Big/ \sim$$

where $\sim$ is the lease equivalence relation satisfying $F(f,\mathsf{id})(x) = F(\mathsf{id},f)(x)$ for all $x \in F(B,A)$ and $f : A \to B$; and α_A is the inclusion $F(A,A) \hookrightarrow \int^{\mathcal{C}} F$.

References

[Ant02] Rosa Antolini. Geometric realisations of cubical sets with connections, and classifying spaces of categories. *Applied Categorical Structures*, 10(5):481–494, 2002.

[CH13] Thierry Coquand and Simon Huber. A model of type theory in cubical sets. 2013.

[Jar02] J. F. Jardine. Cubical homotopy theory: A beginning. Technical report, 2002.