RESEARCH STATEMENT

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1. Introduction

My research interests lie mainly in set theory, in particular in forcing, large cardinals, and combinatorial set theory. The method of forcing is ubiquitous in modern set theory and is used primarily as a tool for proving independence results. It was first developed by Paul Cohen, who used it to prove that the Continuum Hypothesis is independent of the usual axioms of set theory (ZFC) [2]. The technique of forcing is roughly as follows: Given a transitive model M of ZFC and a poset $\mathbb{P} \in M$, one adjoins an object called a $\mathbb{P}$-generic filter, G, to M to create a model of ZFC, $M[G]$, that is in a sense the smallest model of ZFC such that $M \subseteq M[G]$ and $G \in M[G]$ (in non-trivial cases, $G \notin M$). By altering the properties of the poset $\mathbb{P}$, one can determine to an extent what statements are true or false in the generic extension, $M[G]$.

Much of my research involves the use of forcing to produce models of ZFC in which certain combinatorial statements hold. One of the major themes in combinatorial set theory research is the tension that exists between canonical inner models and incompactness phenomena on the one hand and large cardinals and reflection principles on the other. For example, Jensen [11] showed that, if $V = L$, then the combinatorial principle $\square_\kappa$, which is a very strong instance of incompactness, holds for every cardinal $\kappa \geq \omega_1$. On the other hand, a result of Burke and Kanamori (see [15]), implies that, if κ is a strongly compact cardinal, then $\square_\lambda$ fails for every cardinal $\lambda \geq \kappa$. Much work has been done to explore the boundary between these two regimes and to determine exactly which L-like combinatorial principles are consistent with reflection phenomena or the existence of large cardinals (see, for example, [4] or [9]). In some of my research, I investigated covering matrices, combinatorial structures introduced by Viale [18], and proved some implications and consistency results connecting certain types of covering matrices with certain square principles. In the process, I isolated square principles intermediate between the classical $\square_\kappa$ and $\square(\kappa^+)$ and analyzed the system of implications and non-implications that exist among them.

A large part of my work focuses on singular cardinal combinatorics. Combinatorial problems about successors of singular cardinals are inextricably bound up with questions about inner models and large cardinals and often have significant implications for cardinal arithmetic. My results in this area include a PCF-theoretic analysis of a model of Gitik and Sharon [10], some results about the ideal of sets carrying good scales, and a consistency result about bounded stationary reflection at the successors of singular cardinals.

2. Covering Matrices and Squares

A combinatorial structure of particular interest in my research is known as a *covering matrix*.

Definition Let $\theta < \lambda$ be regular cardinals. $\mathcal{D} = \{D(i,\beta) \mid i < \theta, \beta < \lambda\}$ is a *θ-covering matrix for λ* if:

(1) For all $\beta < \lambda$, $\beta = \bigcup_{i<\theta} D(i,\beta)$.
(2) For all $\beta < \lambda$ and all $i < j < \theta$, $D(i,\beta) \subseteq D(j,\beta)$.
(3) For all $\beta < \gamma < \lambda$ and all $i < \theta$, there is $j < \theta$ such that $D(i,\beta) \subseteq D(j,\gamma)$.

$\beta_{\mathcal{D}}$ is the least β such that for all $\gamma < \lambda$ and all $i < \theta$, $\mathrm{otp}(D(i,\gamma)) < \beta$. $\mathcal{D}$ is *normal* if $\beta_{\mathcal{D}} < \lambda$.
$\mathcal{D}$ is *transitive* if, for all $\alpha < \beta < \lambda$ and all $i < \theta$, if $\alpha \in D(i,\beta)$, then $D(i,\alpha) \subseteq D(i,\beta)$.
$\mathcal{D}$ is *uniform* if for all $\beta < \lambda$ there is $i < \theta$ such that $D(j,\beta)$ contains a club in β for all $j \geq i$.

Covering matrices were introduced by Viale in his proof that the Singular Cardinals Hypothesis (SCH) follows from the Proper Forcing Axiom (PFA) [18]. In this work and in subsequent work with Sharon [16], he isolated two important properties which can hold of covering matrices.

Definition Let $\theta < \lambda$ be regular cardinals, and let $\mathcal{D}$ be a θ-covering matrix for λ.

(1) $\mathrm{CP}(\mathcal{D})$ holds if there is an unbounded $T \subseteq \lambda$ such that for every $X \in [T]^\theta$, there are $i < \theta$ and $\beta < \lambda$ such that $X \subseteq D(i, \beta)$.
(2) $\mathrm{S}(\mathcal{D})$ holds if there is a stationary $S \subseteq \lambda$ such that for every family $\{S_j \mid j < \theta\}$ of stationary subsets of S, there are $i < \theta$ and $\beta < \lambda$ such that, for every $j < \theta$, $S_j \cap D(i, \beta) \neq \emptyset$.

The heart of Viale's argument that PFA implies SCH consists of the following.

Theorem 2.1. *(Viale [18]) Let $\lambda > \aleph_2$ be a regular cardinal. PFA implies that $\mathrm{CP}(\mathcal{D})$ holds for every ω-covering matrix $\mathcal{D}$ for λ.*

The assertion that $\mathrm{CP}(\mathcal{D})$ or $\mathrm{S}(\mathcal{D})$ holds for all covering matrices of a particular shape can be seen as a type of reflection principle. I investigated the failure of these principles, i.e. the existence of a covering matrix $\mathcal{D}$ for which $\mathrm{CP}(\mathcal{D})$ and $\mathrm{S}(\mathcal{D})$ fail. Motivated by the case $\theta = \omega_1$, $\lambda = \omega_2$, I first focused on κ-covering matrices for κ^+, where $\kappa \geq \omega_1$ is a regular cardinal. An easy proposition yields that, if $\mathcal{D}$ is a transitive, normal, uniform κ-covering matrix for κ^+, then $\mathrm{CP}(\mathcal{D})$ and $\mathrm{S}(\mathcal{D})$ fail. Results of Sharon and Viale [16] show that sufficiently strong reflection principles preclude the existence of such covering matrices. The following theorem shows that certain square principles actually imply the existence of such covering matrices.

Theorem 2.2. *(L-H [13]) Suppose κ is a regular cardinal and $\square_{\kappa, <\kappa}$ holds. Then there is a transitive, normal, uniform κ-covering matrix for κ^+.*

The following result, proved using a standard argument originally due to Baumgartner [1], shows that the above implication is sharp in the sense that weaker square principles do not imply the existence of the desired covering matrices.

Theorem 2.3. *(L-H [13]) Suppose that, in V, κ is a regular cardinal and there is a measurable cardinal greater than κ. Then there is a forcing extension $V[G]$ in which κ remains a regular cardinal, $\square^*_\kappa$ holds, and there are no transitive, normal, uniform κ-covering matrices for κ^+.*

Also, the implication in 2.2 is not in general reversible.

Theorem 2.4. *(L-H [13]) Suppose that, in V, κ is a regular cardinal that is not strongly inaccessible and there is a measurable cardinal greater than κ. Then there is a forcing extension $V[G]$ in which κ remains a regular cardinal, there is a transitive, normal, uniform κ-covering matrix for κ^+, and $\square_{\kappa, <\kappa}$ fails.*

Turning my attention to covering matrices of a more general shape, I found that certain natural strengthenings of the principle $\square(\lambda)$ become relevant. We thus make the following definition.

Definition Let $\lambda > \omega_1$ be a regular cardinal, and let $\vec{C} = \langle C_\alpha \mid \alpha < \lambda \rangle$ be a $\square(\lambda)$-sequence.

(1) Let $\mu < \lambda$ be a regular cardinal. We say $\vec{C}$ is a $\square^\mu(\lambda)$-sequence if $\{\alpha < \lambda \mid \mathrm{otp}(C_\alpha) = \mu\}$ is stationary in λ. We say $\square^\mu(\lambda)$ holds if there is a $\square^\mu(\lambda)$-sequence.
(2) Let $S \subset \lambda$ be stationary. We say $\vec{C}$ is a $\square(\lambda, S)$-sequence if, for every $\alpha < \lambda$, $\lim(C_\alpha) \cap S = \emptyset$. $\square(\lambda, S)$ holds if there is a $\square(\lambda, S)$-sequence.

The following uses results of Todorčević [17] concerning minimal walks on ordinals.

Proposition 2.5. *(L-H [13]) Suppose $\theta < \lambda$ are infinite, regular cardinals, with $\lambda > \omega_1$. Suppose also that $\square^\theta(\lambda)$ holds. Then there is a θ-covering matrix for λ, $\mathcal{D}$, for which $\mathrm{CP}(\mathcal{D})$ and $\mathrm{S}(\mathcal{D})$ fail.*

The question naturally arises whether the strengthenings of $\square(\lambda)$ defined above are actually stronger than $\square(\lambda)$ (or weaker than $\square_\kappa$, if $\lambda = \kappa^+$.) The following implications are easily obtained.

Proposition 2.6. *(L-H [13]) Let $\mu \leq \nu < \lambda$ be regular cardinals.*

(1) *If $\square^\mu(\kappa)$ holds, then there is a stationary $S \subseteq S^\kappa_\mu$ such that $\square(\kappa, S)$ holds.*
(2) *If there is a stationary $S \subseteq S^\kappa_\omega$ such that $\square(\kappa, S)$ holds, then $\square^\omega(\kappa)$ holds.*
(3) *If $\square^\nu(\kappa)$ holds, then $\square^\mu(\kappa)$ holds.*

In general, if λ is a successor cardinal, these are the only implications that hold. Recall that, if κ is an infinite regular cardinal and $\kappa < \lambda$, then S^λ_κ is the set of $\alpha < \lambda$ such that $\mathrm{cf}(\alpha) = \kappa$.

Theorem 2.7. *(L-H [13]) In V, suppose $\mu \leq \kappa$ are regular cardinals and there is a measurable cardinal greater than κ. Then there is a forcing extension $V[G]$ in which all cardinals and cofinalities $\leq \kappa$ are preserved, $\square^\mu(\kappa^+)$ holds, and $\square_{\kappa, <\kappa}$ fails.*

Theorem 2.8. *(L-H [13]) In V, suppose $\mu < \nu \leq \kappa$ are regular cardinals and there is a measurable cardinal greater than κ. Then there is a forcing extension $V[G]$ in which all cardinals and cofinalities $\leq \kappa$ are preserved and $\square^\mu(\kappa^+)$ holds, but $\square^\nu(\kappa^+)$ fails.*

Theorem 2.9. *(L-H [13]) Suppose $V = L$, κ is a regular, uncountable cardinal, and there is a Mahlo cardinal greater than κ. Then there is a forcing extension $V[G]$ in which all cardinals and cofinaliites $\leq \kappa$ are preserved, there is a stationary $S \subseteq S^{\kappa^+}_\kappa$ such that $\square(\kappa^+, S)$ holds, and $\square(\kappa^+, T)$ fails for every stationary $T \subseteq S^{\kappa^+}_{<\kappa}$.*

Putting this all together, we obtain the following complete picture of implications and non-implications in the case $\lambda = \omega_2$.

$$\begin{tikzcd}
 & \square_{\omega_1} \arrow[dl, bend right] \arrow[dr, bend left] & \\
\square^\omega(\omega_2) \arrow[ur, "/" marking] \arrow[rr, bend right, "/" marking] \arrow[d, shift right] \arrow[drr, "/" marking] & & \square^{\omega_1}(\omega_2) \arrow[ul, bend right, "/" marking] \arrow[ll, bend right] \arrow[d, shift right] \\
\exists \text{ stationary } S \subseteq S^{\aleph_2}_{\aleph_0}(\square(\omega_2, S)) \arrow[u, shift right] \arrow[rr, bend right, "/" marking] \arrow[dr, bend right] & & \exists \text{ stationary } T \subseteq S^{\aleph_2}_{\aleph_1}(\square(\omega_2, T)) \arrow[u, shift right, "/" marking] \arrow[ll, bend right, "/" marking] \arrow[dl, bend right] \\
 & \square(\omega_2) \arrow[ul, "/" marking] \arrow[ur, "/" marking] &
\end{tikzcd}$$

Some questions remain open in the case $\lambda > \omega_2$. For example:

Question Suppose $\mu < \nu < \kappa$ are infinite regular cardinals and there is a stationary $S \subseteq S^{\kappa^+}_\nu$ such that $\square(\kappa^+, S)$ holds. Must there be a stationary $T \subseteq S^{\kappa^+}_\mu$ such that $\square(\kappa^+, T)$ holds?

3. Scales in a model of Gitik and Sharon

In [10], Gitik and Sharon answer two important questions in singular cardinal combinatorics with the following result. Recall that AP_κ stands for the approachability property at κ, which can be seen as a weak square principle.

Theorem 3.1. *(Gitik, Sharon [10]) Suppose that, in V, κ is a supercompact cardinal. Then there is a forcing extension in which κ is a strong limit singular cardinal of countable cofinality, $2^\kappa = \kappa^{++}$, AP_κ fails, and there is $A \subseteq \kappa$ such that there is a very good scale in $\prod A$.*

To obtain this result, Gitik and Sharon first force to make $2^\kappa = \kappa^{+\omega+2}$ while preserving the supercompactness of κ and then force with a diagonal version of supercompact Prikry forcing. Cummings and Foreman [3] showed that, in the model of [10], there is a $B \subseteq \kappa$ such that $\prod B$ carries a bad scale, thus providing another proof of the failure of AP_κ. In [3], Cummings and Foreman ask a number of questions regarding scales in the Gitik-Sharon model, three of which I have addressed in my research.

Let κ be a supercompact cardinal. If $\lambda \leq \kappa$ is inaccessible, let $\mathbb{A}(\lambda)$ be the full-support product of $\mathrm{Add}(\lambda^{+n}, \lambda^{+\omega+2})$ for $n < \omega$, where $\mathrm{Add}(\lambda^{+n}, \lambda^{+\omega+2})$ is the forcing to add $\lambda^{+\omega+2}$-many Cohen subsets of λ^{+n}. Let $\mathbb{P}$ be the iteration with reverse Easton support of $\mathbb{A}(\lambda)$ for all inaccessible $\lambda \leq \kappa$. In $V^{\mathbb{P}}$, let $\mathbb{Q}$ be the diagonal supercompact Prikry forcing at κ defined in [10]. Let $G * H$ be $\mathbb{P} * \mathbb{Q}$-generic over V. Then, in $V[G * H]$, we can completely characterize the scales that exist at κ. We first note that there is a natural ω-sequence of inaccessible cardinals cofinal in κ that is definable from H. We denote this sequence by $\langle \kappa_n \mid n < \omega \rangle$. The very good scale identified in [10] lives in $\prod\limits_{n<\omega} \kappa_n^{+\omega+1}$.

Theorem 3.2. *(L-H [12]) Let $\mathbb{P} * \mathbb{Q}$ be as defined above, and let $G * H$ be $\mathbb{P} * \mathbb{Q}$-generic over V. Then, in $V[G * H]$, the following hold:*

(1) *There is a scale in $\prod_{\substack{n<\omega \\ i \leq n}} \kappa_{n+1}^{+i}$ of length κ^{++} such that every $\alpha < \kappa^{++}$ with $\omega < \mathrm{cf}(\alpha) < \kappa$ is very good.*

(2) *Suppose $\sigma \in {}^\omega\omega$ and, for all $n < \omega$, $\sigma(n) \geq n$. There is a bad scale of length κ^+ in $\prod_{n<\omega} \kappa_n^{+\sigma(n)}$.*

(3) *In $V[G]$, let $\sigma : \omega \to \kappa$ be such that, for all $n < \omega$, $\sigma(n) \geq \omega + 1$. Then, in $V[G * H]$, there is a scale of length κ^{++} in $\prod_{n<\omega} \kappa_n^{+\sigma(n)+1}$ such that every $\alpha < \kappa^{++}$ with $\omega < \mathrm{cf}(\alpha) < \kappa$ is very good.*

By interleaving Levy collapses into the diagonal supercompact Prikry forcing and performing one more Levy collapse at the end, we can obtain a model in which there is a very sharp dividing line between good and bad scales at the relatively small cardinal $\aleph_{\omega^2}$.

Theorem 3.3. *(L-H [12]) Let κ be a supercompact cardinal. Then there is a forcing extension in which $\kappa = \aleph_{\omega^2}$ and, for every $\sigma \in {}^\omega\omega$,*

(1) *If $\sigma(n) < n + 2$ for all but finitely many $n < \omega$, then $\prod_{n<\omega} \aleph_{\omega\cdot(n+1)+\sigma(n)}$ carries a very good scale.*

(2) *If $\sigma(n) \geq n + 2$ for infinitely many $n < \omega$, then $\prod_{n<\omega} \aleph_{\omega\cdot(n+1)+\sigma(n)}$ carries a bad scale.*

The second question of Cummings and Foreman addressed in my research regards the classification of bad points in the bad scales of the Gitik-Sharon model. To state the results, we need the following notion.

Definition Let A be a set of regular cardinals and let $\overrightarrow{f} = \langle f_\alpha \mid \alpha < \lambda \rangle$ be a $<^*$-increasing sequence of functions in $\prod A$. $g \in {}^A\mathrm{On}$ is an *exact upper bound* (*eub*) for $\overrightarrow{f}$ if

- For all $\alpha < \lambda$, $f_\alpha <^* g$.
- For all $h < g$, there is $\alpha < \lambda$ such that $h <^* f_\alpha$.

I was able to show that, in the Gitik-Sharon model, if $\overrightarrow{f}$ is a bad scale of length κ^+ in $\prod_{n<\omega} \kappa_n^{+\sigma(n)}$ for some $\sigma \in {}^\omega\omega$, then there is an inaccessible $\delta < \kappa$ such that, for stationarily many $\beta \in S^{\kappa^+}_{\delta^{+\omega+1}}$, there is an eub, g, for $\overrightarrow{f} \restriction \beta$ such that, for every $n < \omega$, $\mathrm{cf}(g(n)) = \delta^{+n}$. I also analyzed bad points in a bad scale $\overrightarrow{f}$ of length $\aleph_{\omega+1}$ in $\prod_{1 \leq n<\omega} \aleph_n$ from a model of Cummings, Foreman, and Magidor [5], concluding that in this model, there are stationarily many $\beta \in S^{\aleph_{\omega+1}}_{\aleph_1}$ such that there is an $h < f_\beta$ such that the sequence of sets $\langle \{n \mid f_\alpha(n) < h(n)\} \mid \alpha < \beta \rangle$ does not stabilize modulo bounded sets. Both of these situations are different from the behavior of bad points in bad scales derived from Martin's Maximum [7] and from a version of Chang's Conjecture [9].

The last question of Cummings and Foreman which I address is whether, in the case in which the first PCF generator exists, there is a maximal set which carries a good scale. Let us make the following definition.

Definition Suppose κ is a singular cardinal of countable cofinality. Then $I_{gd}[\kappa]$ is the collection of $A \subseteq \kappa$ such that A is a set of regular cardinals and either A is finite or $\mathrm{otp}(A) = \omega$ and $\prod A$ carries a good scale of length κ^+.

$I_{gd}[\kappa]$ is easily seen to be an ideal. The variant of Cummings and Foreman's question which I investigated is whether or not $I_{gd}[\kappa]$ is a P-ideal, i.e., given $\langle A_n \mid n < \omega \rangle$ with $A_n \in I_{gd}[\kappa]$ for all $n < \omega$, whether there is a $B \in I_{gd}[\kappa]$ such that $A_n \subseteq^* B$ for all $n < \omega$, where $\subseteq^*$ denotes containment modulo finite sets. The following result shows that, under certain assumptions on cardinal characteristics of the continuum, things can locally go wrong when trying to find such a B.

Theorem 3.4. *(L-H [12]) Suppose $\mathfrak{b} = \omega_1$. Let $\langle A_n \mid n < \omega \rangle$ be such that, for each n, $A_n \subseteq A_{n+1}$, $A_{n+1} \setminus A_n$ is infinite, and $\bigcup_{n<\omega} A_n = \omega$. There is a sequence of functions $\overrightarrow{f} = \langle f_\alpha \mid \alpha < \omega_1 \rangle$, $<^*$-increasing in ${}^\omega\mathrm{On}$, such that, for every $n < \omega$, $\langle f_\alpha \restriction A_n \mid \alpha < \omega_1 \rangle$ has an eub, g_n, such that $\mathrm{cf}(g_n(i)) = \omega_1$ for all $i < \omega_1$, but for all $B \subseteq \omega$ such that $A_n \subseteq^* B$ for all $n < \omega$, $\langle f_\alpha \restriction B \mid \alpha < \omega_1 \rangle$ does not have an eub.*

This does not provide a counterexample to $I_{gd}[\kappa]$ being a P-ideal, since such a counterexample would require things to go wrong simultaneously at stationarily many points. The consistency of such a counterexample remains open. However, the following result shows that, in a mild forcing extension of V, there are no such counterexamples, and suggests that the problem may have more to do with the structure of the continuum than with PCF-theoretic behavior at higher cardinals.

Theorem 3.5. *(L-H [12]) Let $\langle \mathbb{P}_\gamma \mid \gamma \leq \omega_1 \rangle$ be a finite-support iteration of Hechler forcing. Then, in $V^{\mathbb{P}_{\omega_1}}$, for every singular cardinal κ of countable cofinality, $I_{gd}[\kappa]$ is a P-ideal.*

Question Suppose κ is a singular cardinal of countable cofinality. Must $I_{gd}[\kappa]$ be a P-ideal?

4. Bounded Stationary Reflection

Questions of stationary reflection have played an important role in the investigation of interactions between L-like combinatorial principles, large cardinals, and reflection principles. For example, Jensen, in [11], showed that, if $V = L$ and λ is a regular, uncountable cardinal that is not weakly compact, then there is a stationary $S \subseteq \lambda$ such that S does not reflect. On the other hand, Magidor, in [14], produced, starting with ω-many supercompact cardinals, a model in which every stationary subset of $\aleph_{\omega+1}$ reflects. If a stationary set reflects, it is also of interest to consider the cofinalities of ordinals at which it reflects. This has implications, for example, in the study of square bracket partition relations, where Eisworth has shown [8] that, if μ is a singular cardinal and $\mu^+ \to [\mu^+]^2_{\mu^+}$, then, for every stationary set $S \subseteq \mu^+$ and every regular $\lambda < \mu$, there is $\beta \in S^{\mu^+}_{>\lambda}$ such that S reflects at β. A natural question to ask is whether this is necessarily the case when μ is a singular cardinal and every stationary subset of μ^+ reflects. Let us make the following definition.

Definition Let μ be a singular cardinal. *Bounded stationary reflection* holds at μ^+ if every stationary subset of μ^+ reflects but there is a stationary $S \subseteq \mu^+$ and a $\lambda < \mu$ such that S does not reflect at any ordinal β such that $\mathrm{cf}(\beta) \geq \lambda$.

In joint work with James Cummings, it was shown that, assuming sufficiently many supercompact cardinals, bounded stationary reflection is consistent at the successor of any singular cardinal $\mu > \aleph_\omega$ and that one can achieve bounded stationary reflection at many cardinals simultaneously. More precisely, we have the following.

Theorem 4.1. *(Cummings, L-H [6]) Suppose there is a proper class of supercompact cardinals and GCH holds. Then there is a forcing extension in which, for every singular cardinal $\mu > \aleph_\omega$ that is not a cardinal fixed point, bounded stationary reflection holds at μ^+.*

Question Is it consistent, relative to large cardinals, that bounded stationary reflection holds at the successor of every singular cardinal $\mu > \aleph_\omega$?

5. Current and Future Work

A few open questions remain from my work described above which I plan to continue investigating. For example, given a singular cardinal μ, I would like to understand more fully the structure of the ideal of subsets of μ which carry good scales, in particular to determine whether it is singly generated. I am also interested in exploring other areas of PCF theory, particularly in relation to the following important open question.

Question Is it consistent that there is a set $A \subseteq \omega$ and a scale $\vec{f}$ in $\prod_{n \in A} \aleph_n$ of length $\aleph_{\omega+1}$ with stationarily many bad points of cofinality ω_2?

Also, I would like to produce a truly global version of the result on bounded stationary reflection cited above, namely, a model in which, for every singular $\mu > \aleph_\omega$, bounded stationary reflection holds at μ^+, and to investigate the following major questions which partially motivated the study of bounded stationary reflection.

Question Is it consistent that there is a singular cardinal μ such that $\mu^+ \to [\mu^+]^2_{\mu^+}$? Is it consistent that there is a singular cardinal μ such that μ^+ is a Jónsson cardinal?

Another area I am interested in is the use of Prikry and Radin forcing (forcing notions commonly used to singularize a large cardinal) to prove consistency results regarding combinatorial statements at successors of singular cardinals. For example, it seems possible that such techniques could be used to resolve the following question.

Question (Viale) Suppose $V \subseteq W$ are models of ZFC, κ is an inaccessible cardinal in V, κ is a singular cardinal of cofinality ω_1 in W, and $(\kappa^+)^V = (\kappa^+)^W$. Does $\square_{\kappa,\omega_1}$ necessarily hold in W?

Solutions to problems such as this will likely involve an analysis of a forcing iteration $\mathbb{P} * \mathbb{Q}$, where κ is a large cardinal, $\mathbb{P}$ is forcing which changes the power set of κ, and $\mathbb{Q}$ is Prikry or Radin forcing at κ using a measure or measure sequence extending a measure or measure sequence in the ground model. In particular, it will involve understanding the quotient forcing $\mathbb{P} * \mathbb{Q}/\bar{\mathbb{Q}}$, where $\bar{\mathbb{Q}}$ is Prikry or Radin forcing defined using the ground model measure or measure sequence. Such quotients are currently not well-understood and are of interest in their own right.

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