Inequalities

Varsity Practice 10/6/19

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1 Warm-Up Problems

1. Prove if a, b, c are positive reals and $a + b + c = 1$, then
$$9 \leq 1/a + 1/b + 1/c$$

2. Prove that if $a, b, c \in (0, 1)$, then
$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < 1$$

3. Given $abc = 1$ prove $a + b + c \leq a^2 + b^2 + c^2$.

4. Prove if a, b, c are positive reals, then
$$(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2) \geq 9a^2b^2c^2$$

5. Prove for all positive real a, b, c, d
$$abc + 4abd + acd + 16bcd \geq \frac{64abcd}{a+b+c+d}$$

2 Important Inequalities

AM-GM: If $x_1, ..., x_n$ are positive reals, then
$$\frac{x_1 + x_2 + ... + x_n}{n} \geq (x_1x_2...x_n)^{1/n}$$

Cauchy: If $a_1, ..., a_n, b_1, ..., b_n$ are positive reals, then
$$(a_1 + a_2 + ... + a_n)(b_1 + b_2 + ... + b_n) \geq (\sqrt{a_1b_1} + \sqrt{a_2b_2} + ... + \sqrt{a_nb_n})^2$$

3 Problem Set

1. Let $P(x)$ be a polynomial with positive coefficients. If $P(1/x) \geq 1/P(x)$ holds for $x = 1$, then it holds for all $x > 0$.

2. Prove if a, b, c are positive reals, then
$$\frac{a}{(b+c)^2} + \frac{b}{(a+c)^2} + \frac{c}{(a+b)^2} \geq \frac{9}{4(a+b+c)}$$

3. Prove if a, b, c are positive reals and $abc = 1$, then

$$\frac{a^2}{b+c} + \frac{b^2}{a+c} + \frac{c^2}{a+b} \geq \frac{3}{2}$$

4. Prove if a, b, c, d are positive reals, then

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+d} + \frac{d^2}{d+a} \geq \frac{a+b+c+d}{2}$$

5. Prove if a, b, c, d are positive reals and $a+b+c+d = 8$, then

$$(a + \frac{1}{b+c})^2 + (b + \frac{1}{c+d})^2 + (c + \frac{1}{d+a})^2 + (d + \frac{1}{a+b})^2 \geq 20.25$$

6. Prove if $x_1, ..., x_n$ are positive reals and $x_1 + ... + x_n = 1$, then

$$\Sigma_{i=1}^{n} \left(\frac{x_i^2 + 1}{x_i} \right)^2 \geq n^3 + 2n + 1/n$$

7. Prove if a, b, c are positive reals, then

$$\frac{1}{a(b+1)} + \frac{1}{b(c+1)} + \frac{1}{c(a+1)} \geq \frac{3}{1+abc}$$

8. Let a, b, c, x, y be positive reals. If $abc = 1$, prove

$$\frac{a^3b^3}{ax+by} + \frac{b^3c^3}{bx+cy} + \frac{c^3a^3}{cx+ay} \geq \frac{3}{x+y}$$

9. Let a, b, c, x, y be positive reals. Prove

$$\frac{a}{bx+cy} + \frac{b}{cx+ay} + \frac{c}{ax+by} \geq \frac{3}{x+y}$$

10. Prove if a, b, c are positive reals and $abc = 1$, then

$$\frac{1}{(a+1)^2+b^2+1} + \frac{1}{(b+1)^2+c^2+1} + \frac{1}{(c+1)^2+a^2+1} \leq \frac{1}{2}$$

11. Let a, b, c be positive reals. Prove that:

$$\frac{a^3}{a^2+ab+b^2} + \frac{b^3}{b^2+bc+c^2} + \frac{c^3}{c^2+ac+a^2} \geq \frac{a+b+c}{3}$$