Polynomials

1. Warm-Up

1. Prove that the expression $A(x) = (x-2)^{100}+(x-1)^{50}-1$ is divisible by polynomial $B(x) = x^2-3x+2$.
2. Residue from division of polynomial $P(x)$ on expressions $x-2$ and $x-3$ are 5 and 7 respectively. Find residue from division of polynomial $P(x)$ on expressions x^2-5x+6.
3. Prove identity

$$\frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} + \frac{(x-a)(x-b)}{(c-a)(c-b)} = 1.$$

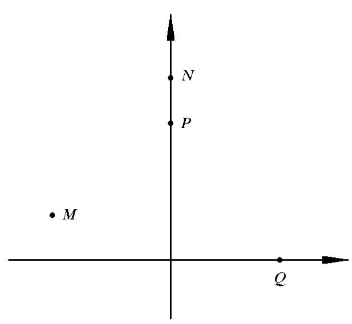
4. Prove that expression $(a+b+c)(ab+bc+ca)-abc$ is divisible by expression $a+b$.

2. Problems

1. For what values of the parameters a, b and c polynomial x^3+ax^2+bx+c is divisible by binomials $x-1$ and $x+2$, but have residue 10 after dividing by binomial $x+1$?
2. Prove identity

$$a^2\frac{(x-b)(x-c)}{(a-b)(a-c)} + b^2\frac{(x-c)(x-a)}{(b-c)(b-a)} + c^2\frac{(x-a)(x-b)}{(c-a)(c-b)} = x^2$$

3. Prove that the expression $a^3(b-c)+b^3(c-a)+c^3(a-b)$ is divisible by the expression $(a-b)(b-c)(c-a)$.
4. Numbers 1 and 5 are roots of polynomial $P(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + 2010$, where all a_i are integers. Is there exist an integer number x_0, such that $P(x_0) = 1000$?
5. Find all pairs of quadratic x^2+ax+b and x^2+cx+d, such that a and b are roots of second quadratic, c and d are roots of first one.
6. Roots of two quadratic x^2+ax+b and x^2+cx+d are negative integer numbers, such that one of them are common. Can it be that values of this quadratics in some positive point will be equal to 20 and 19?
7. Can graphs of polynomials $f(x) = x^3+bx^2+cx+a$ and $g(x) = x^3+ax^2+bx+c$ be such that graph of polynomial $f(x)$ goes through the points M, P and Q, and graph of polynomial $g(x)$ goes through the points M and N (see the picture)?



8. Given a polygon with sides of length $a_1, a_2, \ldots, a_n$ and quadratic $f(x)$ such that $f(a_1) = f(a_2 + ... + a_n)$. Prove that if A is a sum of some of the polygon sides, B is a sum of polygon sides that is left, then $f(A) = f(B)$.

9. Prove that for any polynomial $P(x)$ with natural coefficients there exists infinite number of positive integers n, such that each $P(n)$ has the same sum of digits.

3. Bonus

1. Polynomial $P(x) = x^3 + ax^2 + bx + c$ has three different real roots, and polynomial $P(Q(x))$ has not real roots, where $Q(x) = x^2 + x + 2001$. Prove that $P(2001) > \frac{1}{64}$.

2. There are polynomials $f(x)$ and $g(x)$ with integer nonzero coefficients, m is the greatest coefficient of polynomial f. For some natural numbers $a < b$ it is true that $f(a) = g(a)$ and $f(b) = g(b)$. Prove that if $b > m$ than polynomials f and g are equal.