Combinatorics Review 2

David Altizio

1. (HMMT 2005 Combinatorics #2) How many nonempty subsets of $\{1, 2, 3, \ldots, 12\}$ have the property that the sum of the largest element and the smallest element is 13?

2. (HMMT 2013 Combinatorics #2) If Alex does not sing on Saturday, then she has a 70% chance of singing on Sunday; however, to rest her voice, she never sings on both days. If Alex has a 50% chance of singing on Sunday, find the probability that she sings on Saturday.

3. (HMMT 2003 Combinatorics #6) In Porter A19D, 44 students are seated in 9 rows of 5 chairs.[1] The chair in the back-left corner is unoccupied. C.J. decides to reassign the seats such that each student will occupy a chair adjacent to his/her present one (i.e. move one desk forward, back, left or right). In how many ways can this reassignment be made?

4. (CMIMC 2016 Combinatorics #4) Let $\mathcal{S}$ be a regular 18-gon, and for two vertices in $\mathcal{S}$ define the *distance* between them to be the length of the shortest path along the edges of $\mathcal{S}$ between them (e.g. adjacent vertices have distance 1). Find the number of ways to choose three distinct vertices from $\mathcal{S}$ such that no two of them have distance 1, 8, or 9.

5. (HMMT 2008 Combinatorics #4) How many rearrangements of the letters of "HMMTHMMT" do not contain the substring "HMMT"? (For instance, one such arrangement is HMMHMTMT.)

6. (Math Kangaroo 2012) In the center of every cell of a 5×5 board stands one kangaroo. Suddenly, a thunder strikes, and each kangaroo is startled so that it jumps over the side of its cell into a neighboring cell, possibly joining one or more other kangaroos there. What is the greatest possible number of cells that are now empty?

7. (BMT 2017 Discrete #6) Let $S = \{1, 2, \ldots, 6\}$. How manny functions $f : S \to S$ are there such that $f^5(s) = 1$ for all $s \in S$? Note that
$$f^5(s) = f(f(f(f(f(s)))))$$

8. (CMIMC 2016 Team #9, simplified) For how many permutations π of $\{1, 2, \ldots, 6\}$ does there exist an integer N such that
$$N \equiv \pi(i) \pmod{i} \text{ for all integers } 1 \le i \le 6?$$

9. (NIMO 26 #3) Six competitors enter a round-robin tournament (each pair of players plays exactly one round). Each round between two players is equally likely to result in a win for a given player, a loss for that player, or a tie. The results of the tournament are *nice* if for all triples of distinct players (A, B, C),

   - If A beat B and B beat C, then A also beat C;
   - If A and B tied, then either C beat both A and B, or C lost to both A and B.

   Find the probability that the results of the tournament are nice.

[1] If only we could fit this many chairs in real life!

10. (Red MOP 2006) There are 51 senators in a senate. The senate needs to be divided into n committees such that each senator is on exactly one committee. Each senator hates exactly three other senators. (If senator A hates senator B, then senator B does *not* necessarily hate senator A.) Find the smallest n such that it is always possible to arrange the committees so that no senator hates another senator on his or her committee.