Algebra Review 1

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1 JV Mock Individual Round

1. Compute all real valus x for which $4x^2 + 4x^4 + 4x^6 + \cdots = 5$
2. (AMC10 2003) What is the sum of the reciprocals of the roots of the equation
$$\frac{2003}{2004}x + 1 + \frac{1}{x} = 0?$$
3. (AMC12 2014) The first three terms of a geometric progression are $\sqrt{3}$, $\sqrt[3]{3}$, and $\sqrt[6]{3}$. What is the fourth term?
4. (PUMaC 2010) Assume that $f(a+b) = f(a) + f(b) + ab$ and that $f(75) - f(51) = 1230$. Find $f(100)$.
5. Find the sum of the coefficients of the polynomial $(57x - 55)^4$.
6. (PUMaC 2011) A polynomial p can be written as
$$p(x) = x^6 + 3x^5 - 3x^4 + ax^3 + bx^2 + cx + d.$$
Given that all roots of $p(x)$ are equal to either m or n where m, n are integers, compute $p(2)$.
7. Let the recurrence a_n satisfy $a_n = \frac{1+a_{n-1}}{a_{n-2}}$ where $a_0 = 4$ and $a_1 = 7$. Compute a_{209}.
8. (Math League HS 2011-2012) If a is real, what is the only real number that could be a multiple root of $x^3 + ax + 1 = 0$?
9. Solve the recurrence $a_n = 2a_{n-1} + 3$ for $n \geq 2$, where $a_1 = 1$.
10. The four zeros of the polynomial $x^4 + jx^2 + kx + 225$ are distinct real numbers in arithmetic progression. Compute the value of j.

2 Varsity Mock Individual Round

1. Solve the recurrence $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ where $a_0 = 5$, $a_1 = -1$, and $a_2 = 7$.

2. (PUMaC 2010) Assume that $f(a+b) = f(a) + f(b) + ab$ and that $f(75) - f(51) = 1230$. Find $f(100)$.

3. Find the sum of the coefficients of the polynomial $(57x - 55)^4$.

4. (PUMaC 2011) A polynomial p can be written as

$$p(x) = x^6 + 3x^5 - 3x^4 + ax^3 + bx^2 + cx + d.$$

Given that all roots of $p(x)$ are equal to either m or n where m, n are integers, compute $p(2)$.

5. Let the recurrence a_n satisfy $a_n = \frac{1+a_{n-1}}{a_{n-2}}$ where $a_0 = 4$ and $a_1 = 7$. Compute a_{209}.

6. (PUMaC 2011) Suppose the polynomial $x^3 - x^2 + bx + c$ has real roots a, b, c. What is the square of the minimum value of abc?

7. Solve the recurrence $a_n = 2a_{n-1} + 3$ for $n \geq 2$, where $a_1 = 1$.

8. The four zeros of the polynomial $x^4 + jx^2 + kx + 225$ are distinct real numbers in arithmetic progression. Compute the value of j.

9. Suppose that p, q are two-digit prime numbers such that $p^2 - q^2 = 2p + 6q + 8$. Compute the largest possible value of $p + q$.

10. (AIME 2007) A sequence is defined over non-negative integral indexes in the following way: $a_0 = a_1 = 3$, $a_{n+1}a_{n-1} = a_n^2 + 2007$. Find the greatest integer that does not exceed $\frac{a_{2006}^2 + a_{2007}^2}{a_{2006}a_{2007}}$.

3 Team Round

1. (Math League HS 2011-2012) There are an infinite number of polynomials P for which $P(x+5)-P(x)=2$ for all x. What is the least possible value of $P(4)-P(2)$?

2. (AMC10 2004) Let $a_1, a_2, \cdots,$ be a sequence such that $a_1=1$ and $a_{2n}=n\cdot a_n$ for any positive integer n. What is the value of $a_{2^{100}}$?

3. (CMIMC 2018) Let $P(x)=x^2+4x+1$. What is the product of all real solutions to the equation $P(P(x))=0$?

4. Let p and q be real numbers with $|p|<1$ and $|q|<1$ such that
$$p+pq+pq^2+pq^3+\cdots=2 \qquad \text{and} \qquad q+qp+qp^2+qp^3+\cdots=3.$$
What is $100pq$?

5. (CMIMC 2017) Suppose $P(x)$ is a quadratic polynomial with integer coefficients satisfying the identity
$$P(P(x))-P(x)^2=x^2+x+2016$$
for all real x. What is $P(1)$?

6. (HMMT ??) Let $Q(x)=x^2+2x+3$, and suppose that $P(x)$ is a polynomial such that
$$P(Q(x))=x^6+6x^5+18x^4+32x^3+35x^2+22x+8.$$
Compute $P(2)$.

7. (CMIMC 2018) Suppose P is a cubic polynomial satisfying $P(0)=3$ and
$$(x^3-2x+1-P(x))(2x^3-5x^2+4-P(x))\leq 0$$
for all $x\in\mathbb{R}$. Determine all possible values of $P(-1)$.

8. (OMO, Ray Li) Let $a_1, a_2, \ldots$ be a sequence defined by $a_1=1$ and for $n\geq 1$,
$$a_{n+1}=\sqrt{a_n^2-2a_n+3}+1.$$
Find a_{513}.

9. (AMC12 2005) Let $P(x)=(x-1)(x-2)(x-3)$. For how many polynomials $Q(x)$ does there exist a polynomial $R(x)$ of degree 3 such that $P(Q(x))=P(x)\cdot R(x)$?

10. (AoPS) Suppose $(a_n)_{n\geq 1}$ is a sequence of real numbers satisfying
$$a_{2n}=4a_n=2a_{2n-1}+\frac{1}{4}$$
for all $n\geq 1$. Compute the sum $a_1+\cdots+a_{31}$.