Polynomials

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1 JV Warm-Up

1. The area of a rectangle is 18 and the length of a diagonal is 8. Find the perimeter.
2. Given that $x + y + z = 7$ and $x^2 + y^2 + z^2 = 10$, compute $xy + yz + xz$.
3. A parabola with vertex $(\frac{1}{4}, -\frac{9}{8})$ has equation $ax^2 + bx + c = 0$, where $a > 0$ and $a + b + c$ is an integer. Compute the smallest possible value of a.

2 JV Problems

1. For each of the following polynomials, find the product and sum of the roots. Do you notice a pattern?
   (a) $x^2 + x - 6$
   (b) $x^2 + 14x + 42$
   (c) $8x^2 - 10x + 3$
   (d) $x^3 - 4x^2 - 9x + 36$
2. Let r, s, and t be the roots of $f(x) = x^3 - 4x^2 - 7x + 10$. Find $r^2 + s^2 + t^2$.
3. Let r, s, and t be the roots of $f(x) = 2x^3 - 4x^2 + 3x - 9$. Find $(2 - r)(2 - s)(2 - t)$.
4. Let a and b be the roots of the equation $x^2 - mx + 2 = 0$. Suppose that $a + \frac{1}{b}$ and $b + \frac{1}{a}$ are the roots of the equation $x^2 - px + q = 0$. What is q?
5. Suppose the roots of $x^3+3x^2+4x-11 = 0$ are a, b, and c, and the roots of $x^3+rx^2+sx+t = 0$ are $a + b, b + c$, and $c + a$. Find r and t.
6. Three of the roots of $x^4 + ax^2 + bx + c = 0$ are $3, -6, 5$. Find the value of $a + b + c$.
7. A quadratic equation $ax^2 - 2ax + b = 0$ has two real solutions. What is the average of these two solutions?
8. If $P(x)$ is a polynomial, and we have that $x^{20} + 13x^{12} - 4x^3 + 9 = (x^2 - 3x + 17) \times P(x)$ for all x, compute the sum of the coefficients of $P(x)$.

3 JV Challenge Problems

1. Find the sum of all the roots of the equation $x^{2001} + (\frac{1}{2} - x)^{2001} = 0$.
2. Let $P(x)$ be a fifth-degree polynomial with integer coefficients and that has at least one integer root. If $P(2) = 13$ and $P(10) = 5$, compute a value of x that must satisfy $P(x) = 0$.

4 Varsity Warm-Up

1. (Vieta's Formula). Let the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ have roots $r_1, r_2, \ldots, r_n$, listed with multiplicity. Determine (and prove) a formula for the sum of all possible k-fold products of the roots.

2. (Newton's Sums). Consider a polynomial $P(x)$ of degree n,

   $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Let $P(x) = 0$ have roots $x_1, x_2, \ldots, x_n$. Define the following sums:

$$P_1 = x_1 + x_2 + \cdots + x_n$$
$$P_2 = x_1^2 + x_2^2 + \cdots + x_n^2$$
$$\vdots$$
$$P_k = x_1^k + x_2^k + \cdots + x_n^k$$

   Prove that:

$$\begin{array}{ccccc} & a_n P_1 + & a_{n-1} & & = 0 \\ a_n P_2 + & a_{n-1}P_1 + & 2a_{n-2} & & = 0 \\ a_n P_3 + & a_{n-1}P_2 + & a_{n-2}P_1 + & 3a_{n-3} & = 0 \end{array}$$

3. (Descartes' Law of Signs). Given a polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, determine and prove an upper bound on the number of positive real roots of P based on its coefficients. For extra fun, also prove an upper bound on the number of negative real roots.

5 Varsity Problems

1. If $P(x)$ is a polynomial, and we have that $x^{20} + 13x^{12} - 4x^3 + 9 = (x^2 - 3x + 17) \times P(x)$ for all x, compute the sum of the coefficients of $P(x)$.

2. Three of the roots of $x^4 + ax^2 + bx + c = 0$ are $3, -6, 5$. Find the value of $a + b + c$.

3. Let w, x, y, z be the roots of the polynomial $P(a) = 5a^4 - 7a^3 + a^2 - a + 9$. Compute $\frac{1}{w+2}\frac{1}{x+2}\frac{1}{y+2}\frac{1}{z+2}$.

4. (ARML 1985) Let $P(x)$ be a fifth-degree polynomial with integer coefficients and that has at least one integer root. If $P(2) = 13$ and $P(10) = 5$, compute a value of x that must satisfy $P(x) = 0$.

5. (AIME 2001) Find the sum of all the roots of the equation $x^{2001} + (\frac{1}{2} - x)^{2001} = 0$.

6. Determine the sum of the squares of the roots (i.e. P_2 from the hw) of the polynomial $P(x) = 3x^3 + 6x^2 - x + 8$. Can you find a closed form using the coefficients of any polynomial P?

7. (ARML 2006) Determine the sum of the y-coordinates of the four points of intersection of $y = x^4 - 5x^2 - x + 4$ and $y = x^2 - 3x$.

8. Let x_1, x_2, x_3, x_4 be real numbers such that $x_1 + x_2 + x_3 + x_4 = 0$ and $x_1^7 + x_2^7 + x_3^7 + x_4^7 = 0$. Compute the value of $x_1(x_1 + x_2)(x_1 + x_3)(x_1 + x_4)$.

9. (ARML 1983) Let a, b, and c be the sides of triangle ABC. If a^2, b^2, and c^2 are the roots of the equation $x^3 - Px^2 + Qx - R = 0$ (where P, Q, R are constants), express

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$$

in terms of one or more of the coefficients P, Q, R.

10. (PUMaC 2006) Find all pairs of real numbers (a, b) so that there exists a polynomial $P(x)$ with real coefficients and $P(P(x)) = x^4 - 8x^3 + ax^2 + bx + 40$.

11. (PUMaC 2006) Suppose that $P(x)$ is a polynomial with the property that there exists another polynomial $Q(x)$ to satisfy $P(x)Q(x) = P(x^2)$. $P(x)$ and $Q(x)$ may have complex coefficients. If $P(x)$ is a quintic with distinct complex roots $r_1, \ldots, r_5$, find all possible values of $|r_1| + \cdots + |r_5|$.

6 Varsity Challenge Problems

1. (Putnam 2003). Let $f(z) = az^4 + bz^3 + cz^2 + d^z + e = a(z - r_1)(z - r_2)(z - r_3)(z - r_4)$ where a, b, c, d, e are integers, and $a \neq 0$. Show that if $r_1 + r_2$ is a rational number and $r_1 + r_2 \neq r_3 + r_4$, then $r_1 r_2$ is a rational number.

2. (Mock Putnam Exam UTK 2001). Let $f(x)$ be a polynomial with integer coefficients. Suppose there exist distinct integers a, b, c such that $f(a) = f(b) = f(c) = 2000$. Show that there is no integer d such that $f(d) = 2001$.