Equivalence Relations

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1 Introduction

Before we start the problems, we need a few definitions.

Definition 1. Let X be any set. A *relation* R on X is a subset of $X \times X$, i.e. it is a collection of ordered pairs of elements in X. We sometimes write xRy to denote that $(x, y) \in R$.

Example 1. Let $X = \{1, 2, 3\}$. Then $\{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$ is a relation on X, commonly known as '$\leq$'.

Definition 2. Let R be a relation on a set X. We say that R is

- *reflexive* if $(x, x) \in R$ for all $x \in X$;
- *symmetric* if $(x, y) \in R$ implies $(y, x) \in R$ for all $x, y \in X$;
- *transitive* if $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$ for all $x, y, z \in X$.

Example 2. $\leq$, as defined above, is reflexive and transitive but not symmetric.

Definition 3. We say that a relation $\sim$ on a set X is an *equivalence* relation if it is reflexive, symmetric, and transitive. We write $x \sim y$ to denote that x and y are related under $\sim$.[1]

Definition 4. Let $\sim$ be an equivalence relation on a set X. Suppose $Y \subset X$ is such that

- for all $a, b \in Y$, $a \sim b$, and
- for all $a \in Y$ and $b \notin Y$, $a \not\sim b$.

Then Y is said to be an *equivalence class* of X by $\sim$.

2 Problems

1. Determine whether the following relations are equivalence relations on the given set S. If the relation is in fact an equivalence relation, describe its equivalence classes.

   (a) $S = \mathbb{N} \setminus \{0, 1\}$; $(x, y) \in R$ if and only if $\gcd(x, y) > 1$.

   (b) $S = \mathbb{R}$; $(a, b) \in R$ if and only if
   $$a^2 + a = b^2 + b.$$

   (c) $S = \mathbb{R}$; $(x, y) \in R$ if and only if there exists $n \in \mathbb{Z}$ such that $x = 2^n y$.

   (d) (MIT 6.042) $S = P$, where P is the set of all people in the world today; $(x, y) \in R$ if and only if x is at least as tall as y.

[1] The change in notation is admittedly weird, but it is conventional, so we will stick to it.

(e) (BYU) $S = \mathbb{Z}$; $(x, y) \in R$ if and only if $2x + 5y \equiv 0 \pmod 7$.

2. Suppose a relation R on a set S is *antisymmetric* if the following holds: whenever x and y in S satisfy xRy and yRx, then $x = y$. (For reference, an example of such a relation is the $\le$ relation on $\mathbb{R}$.) If an equivalence relation $\sim$ on a set S is also antisymmetric, then what can we say about $\sim$?

3. Let $\sim_1$ and $\sim_2$ be two equivalence relations on the same set S.

   (a) Is the relation $\sim$ on S defined by
   $$x \sim y \text{ if } x \sim_1 y \text{ and } x \sim_2 y$$
   an equivalence relation?

   (b) Is the relation $\sim$ on S defined by
   $$x \sim y \text{ if } x \sim_1 y \text{ or } x \sim_2 y$$
   an equivalence relation?

4. It may not be so obvious that equivalence classes of an equivalence relation are nice to work with. With this in mind, let $Y_1, \ldots, Y_\ell$ be subsets of some set X. Prove that the following are equivalent.

   - There exists an equivalence relation $\sim$ on X with $Y_1, \ldots, Y_\ell$ being its equivalence classes;
   - $Y_1, \ldots, Y_\ell$ forms a partition of X, i.e. $Y_i \cap Y_j = \varnothing$ for all $1 \le i < j \le \ell$ and
   $$X = Y_1 \cup Y_2 \cup \cdots \cup Y_\ell.$$

5. (Tripos 2011) Write down an equivalence relation on the positive integers that has exactly four equivalence classes, of which two are infinite and two are finite.

6. For all $n \ge 0$, let B_n denote the number of equivalence relations on the set $\{1, 2, \ldots, n\}$, where here we define $B_0 = 1$. Show that B_n is finite by giving an explicit upper bound in terms of n.

7. Fix $n \ge 3$. Let C_n denote the number of equivalence relations $\sim$ on the set $\{1, 2, \ldots, n\}$ such that $1 \sim 2$. Let D_n denote the number of equivalence relations $\sim$ on the set $\{1, 2, \ldots, n\}$ such that $1 \not\sim 2$. Determine, with proof, which of C_n, D_n is larger.

8. For all $n \ge 0$, denote by B_n the number from Problem 6.

   (a) Show that
   $$B_{n+1} = \sum_{k=0}^{n} B_k \binom{n}{k}$$
   for all $n \ge 0$.

   (b) Show that
   $$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}$$
   for all $n \ge 0$. You may take the $n = 0$ case for granted.

3 Selected solutions (sketched)

5. We can specify just the equivalence classes. For example, $\{1\}$, $\{2\}$, $\{$odds greater than 1$\}$, $\{$evens greater than 2$\}$ does the job.

6. A relation is defined as the a subset of $X \times X$ where $X = \{1, 2, \ldots, n\}$. This set has n^2 elements, so it has 2^{n^2} subsets, which gives a bound on the number of equivalence relations.

7. D_n is larger. Take any equivalence $\sim$ relation in C_n. Define a new equivalence relation $\sim'$ by simply removing the element 1 from its equivalence class in $\sim$, and placing it in its own equivalence class. Now $1 \not\sim' 2$, and we clearly get a distinct $\sim'$ for distinct $\sim$. Thus $C_n \leq D_n$ is at least as large. Also note that D_n includes any equivalence relation in which 1 is not in an equivalence class by itself but is also not in the same class is 2, but that no $\sim$ such that $1 \sim 2$ maps to this equivalence relation. Since $n \geq 3$ there is at least one such class, so $C_n < D_n$.

8. (a) For any equivalence relation $\sim$ on $\{1, \ldots, n, n+1\}$, let k be the number of elements $i \in \{1, 2, \ldots, n\}$ such that $i \not\sim n+1$. k can range from 0 to n, for each fixed k there are $\binom{n}{k}$ ways to choose the k elements that are not equivalent to $n+1$, and B_k ways to define the equivalence relation on these k elements.

   (b) Use the well-known fact that $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ for the base case, and then apply induction using part (a).