Induction

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1 What is induction?

Induction essentially formalizes “keep going” arguments (i.e. start doing this process and eventually you get the result you want). It is often used to determine properties of recursive sequences. The skeleton of a typical induction proof looks like:

“The statement you want to prove in terms of n”

Base Case: Proving the small case where $n = n_0$

Inductive Step: (Inductive Hypothesis) Assume that the result holds for some $n \in \mathbb{Z}, n \geq n_0$. *Alternatively, you can assume that result holds for all $k \in \mathbb{Z}$ such that $n_0 \leq k \leq n$. This is often called* ***Strong Induction***
Using this assumption, prove that the result holds for $n + 1$.

Conclude by induction that the result holds for all $n \in \mathbb{Z}, n \geq n_0$.

2 Examples

1. Given that every n-gon has a diagonal which lies completely within it, show that for any integer $n \geq 3$, any n-gon (not just convex!) can be triangulated by diagonals which lie within the n-gon.

2. $\sum\limits_{i=1}^{n} i = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$

3. All Horses are the Same Color
Base case: When we have 1 horse, all horses in this group are the same color.
Inductive step: Assume that whenever there is a group of n horses, they are all the same color. Now suppose we have $n + 1$ horses. Ignoring the last horse, we have a group of n horses, so they must all be the same color, by our induction hypothesis. Ignoring the first horse, we have a group of n horses, so similarly, they must all be the same color. Thus, the first horse is the same color as all the horses in the middle, which are the same color as the last horse, so all $n + 1$ horses are the same color.
Therefore, by induction, all horses are the same color.

Many of the below problems come from Arthur Engel's Problem Solving Strategies

3 JV Problems

1. Show by induction that for all positive integers n, the sum of the **odd** positive integers up to $2n-1$ is equal to n^2.
   $1+3+5+7+...+(2n-1) = \sum_{i=1}^{n}(2i-1) = n^2$.
   Bonus: Can you give another explanation for this fact?

2. Prove that all numbers of the form $1007, 10017, 100117, \ldots$ are divisible by 53.

3. Prove that all numbers of the form $7^n - 1$ are divisible by 6 without using modular arithmetic.

4. Prove for any integer $n \geq 3$, that the sum of the degrees of any n-gon is $180(n-2)$.

5. The Fibonacci sequence is defined by $F = 1, 1, 2, 3, 5, 8, 13, 21, 34, ...$ where F_n is the n^{th} number in the sequence, where $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Show that each term of the sequence can be computed explicitly by

$$F_n = \frac{\phi^n - \psi^n}{\phi - \psi}$$

   where $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$.

4 Varsity Problems

1. Show that for all positive integers n, the number $4^n + 15n - 1$ is divisible by 9.

2. The Fibonacci sequence is defined by $F = 1, 1, 2, 3, 5, 8, 13, 21, 34, ...$ where F_n is the n^{th} number in the sequence, where $F_0 = 0, F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Show that $F_{m+n} = F_{m-1}F_n + F_mF_{n+1}$. Can you do it with induction and with a clever counting argument?

3. Prove that $3^{n+1} \mid 2^{3^n} + 1$ for all integers $n \geq 0$.

4. Alex and C.J. each have a number on their forehead. They can see each others' numbers, but not their own. Misha walks by, and as he's walking away, he correctly states "your two numbers are consecutive positive integers less than 10^{10}." Then Alex and C.J. alternate turns, each one either stating the number on his forehead (he must be 100% sure) or admitting that he does not know the number on his forehead. Alex goes first, and the play continues until one person wins by figuring out his number, or both simultaneously drop dead from exhaustion after 10^{100} rounds. What happens?

5. Consider the sequence a_n recursively defined by $a_0 = 0$, $a_{n+1} = \frac{1}{2\lfloor a_n \rfloor - a_n + 1}$. Show that every non-negative rational number appears as an element of a_n exactly once.

6. Prove that if $a, b, q = \frac{a^2+b^2}{ab+1}$ are integers ≥ 0, then $q = \gcd(a,b)^2$. *Hint: Induct on ab*

5 Homework

Please solve and write up your solutions to the problems to submit at the next practice (April 15^{th})!

5.1 Junior Varsity

1. Show that the sum of the first n squares is equal to $\frac{n(n+1)(2n+1)}{6}$, i.e.

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

2. Prove that for all positive integers n, the number $4^n + 15n - 1$ is divisible by 9.

3. You are given $2n$ points in the plane such that no 3 points are in a line. You draw line segments connecting points such that no triangle is formed with the points as vertices. Show that you drew at most n^2 lines.

5.2 Varsity

1. Show that for every integer $k \geq 8$, it can be written in the form $k = 3m + 5n$ for some positive integers m, n.

2. Let F_n denote the n-th Fibonacci number. Let m, n be positive integers such that $n \geq 3$. Show that if $m \mid n$, then $F_m \mid F_n$. (The Fibonacci numbers are defined by $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$.)

3. Prove that every positive integer has a base-10 expansion. Trickier: prove that every positive integer can be written as a sum of distinct powers of -2.