Proof Techniques

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1 Introduction

To successfully problem solve in mathematics, we must understand different methods we can use to approach problems and how to communicate these reasonings. Last week, we mostly used direct proofs, in which we directly used our assumptions to prove the desired results. We also noted that to show that a statement is false, we can sometimes produce a counterexample to do so. This week, we introduce techniques where we consider what happens if our conclusion were false.

1.1 Contrapositive

We show that if the conclusion is false, then the original assumption cannot hold. Note that this is actually equivalent to the direct proof—if the original assumption held but we could not reach the desired conclusion, then the statement would be false!

1. For each integer n, if n^2 is odd, then n is odd.
2. For real numbers p and q, if p is irrational, then q is irrational or $p+q$ is irrational.

Note that in the second example, we must consider when the statement "q is irrational or $p+q$ is irrational" is false. As a logical "or" means that at least one of these things happens, for this to be false, we need both "q is irrational" and "$p+q$ is irrational" to be false. (What happens if we had an "and" instead?)

1.2 Contradiction

We show that if original statement is false, then we can derive nonsense (eg. $0 = 1, x^2 < 0$). To do this, we often assume that the original assumption holds and the conclusion is false, and then derive nonsense.

1. Show that there are infinitely many primes.
2. Show that $\sqrt{2}$ is irrational.

2 Exercises

2.1 Junior Varsity

1. Show that if you put $2n+1$ objects in 2 boxes, then some box has more than n objects. Can you generalize this for k boxes?
2. Given 5 points inside a square. Show that there is a pair whose distance is at most $\sqrt{2}$.
3. Show that if r is irrational, then $r^{\frac{1}{5}}$ is irrational.
4. Is there an integer n such that the sum of the digits of n^2 is 3?
5. Show that a pentagon has at least one obtuse angle.

2.2 Varsity

1. Prove or give a counterexample: if $x^5 + 7x^3 + 5x \geq x^4 + x^2 + 8$, then $x \geq 0$.
2. Is there a triangle whose altitudes have length at least 2 and whose area is 1?
3. Show that $\frac{21n+4}{14n+3}$ is irreducible for every integer n.
4. Whenever trick-or-treaters come to her door on Halloween, a mathematician makes them choose two positive real numbers x and y. She lets s be the smallest of $x, y + 1/x$, and $1/y$ and gives them s pounds of candy. What is the largest possible value of s?
5. Show that the cube roots of three distinct prime numbers cannot be three terms (not necessarily consecutive) of an arithmetic progression.

3 Homework

Please solve and write up your solutions to the problems to submit at the next practice (April 8^{th})!

3.1 Junior Varsity

1. Are there integers a, b such that $15a + 33b = 2$?
2. Show that if n is composite, then it has a prime factor p such that $p \leq \sqrt{n}$.
3. Show that $a^2 - 4b = 2$ has no integer solutions.

3.2 Varsity

1. Let $a \leq b \leq c$ be positive integers and they are the lengths of a right triangle. Show that $3 \mid ab$.
2. Let S be the set of rational numbers r such that $r = q^q$ for some nonzero rational number q. What is S? Note that you have to show not only which numbers are in S, but also which ones are not.
3. The integers from 2 to 1000 are written on the blackboard. The students in school play the following game. Each student in turn picks a number on the blackboard and erases it together with all of its multiples. The game ends when only primes are left written on the board. What is the smallest number of students that need to play before the game ends?