Number Theory 3

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Problems

These problems are from "104 Number Theory Problems" by D. Andrica, T. Andreescu, Z. Feng.

1. . What is the largest positive integer n for which $n^3 + 100$ is divisible by $n + 10$?
2. . Those irreducible fractions! (1) Let n be an integer greater than 2. Prove that among the fractions

   $$\frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}$$

   even number are irreducible.

   (2) Show that the fraction

   $$\frac{12n+1}{30n+2}$$

   is irreducible for all positive integers n.
3. Call a number prime looking if it is composite but not divisible by 2, 3, or 5. The three smallest prime-looking numbers are 49, 77, and 91. There are 168 prime numbers less than 1000. How many prime-looking numbers are there less than 1000?
4. Let $n = 2^{31}3^{19}$. How many positive integer divisors of n 2 are less than n but do not divide n?
5. Compute the sum of the greatest odd divisor of each of the numbers 2006, 2007, . . . , 4012.
6. Let m and n be positive integers such that

   $$lcm(m, n) + gcd(m, n) = m + n.$$

   Prove that one of the two numbers is divisible by the other.
7. Compute the sum of all numbers of the form a/b, where a and b are relatively prime positive divisors of 27000.
8. We'll solve this IMO Shortlist problem using following steps :

   (IMO 2003 shortlist) Determine the smallest positive integer k such that there exist integers $x_1, x_2, \dots, x_k$ with ${x_1}^3 + {x_2}^3 + \cdots + {x_k}^3 = 2002^{2002}$.

   (i) Find all possible values of x^3 on mod 9

   (ii) Find 2002^{2002} on mod 9.

(iii) Prove that 2002^{2002} can never be represented as a sum of 3 cubes using mod 9 remainders.

(iv) Represent 2002 as sum of cube 4 positive integer

(v) Represent 2002^{2002} as sum of 4 cubes.

9. Let S be full residue class on mod p, for a prime number p (i.e. the set $\{0, 1, 2, ..., p-1\}$). Let a be any number s.t. $1 \leq a \leq p-1$. Prove that $aS = \{0, a, 2a, 3a, ..., (p-1)a\}$ is also a full residue class on mod p.
10. Prove Fermat's little theorem, i.e. for $1 \leq a \leq p-1$, following should hold $a^{p-1} \equiv 1 \bmod p$.