Length Chasing

Length chasing is not as well defined as angle chasing; I consider it to be the act of solving for lots of lengths of segments or using metric relationships to solve problems. While these problems may not be beautiful, they are certainly important.

1 Problems

Problems are ordered in terms of difficulty. Challenging problems are marked with a ★.

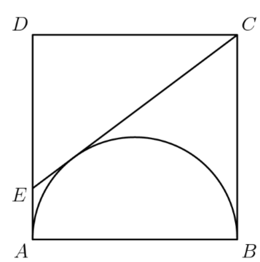
1. [CMIMC 2016, Own] Let $\triangle ABC$ be an equilateral triangle and P a point on $\overline{BC}$. If $PB = 50$ and $PC = 30$, compute PA.

2. [Own] Rectangle $ABCD$ has $AB = 5$ and $BC = 12$. CD is extended to a point E such that $\triangle ACE$ is isosceles with base AC. What is the area of triangle ADE?

3. [Mandelbrot 2010-2011] Two points A and B are located on a circle with center O and radius r. The tangents to the circle at A and B intersect at a point P. If $AB = 12$, $OP = 13$, and $r < 10$, then compute r.

4. [AIME 2006] In quadrilateral $ABCD, \angle B$ is a right angle, diagonal $\overline{AC}$ is perpendicular to $\overline{CD}$, $AB = 18, BC = 21$, and $CD = 14$. Find the perimeter of $ABCD$.

5. [AMC 10A 2004] Square $ABCD$ has side length 2. A semicircle with diameter AB is constructed inside the square, and the tangent to the semicircle from C intersects side AD at E. What is the length of CE?



6. [Math Prize for Girls 2011] Let $\triangle ABC$ be a triangle with $AB = 3$, $BC = 4$, and $AC = 5$. Let I be the center of the circle inscribed in $\triangle ABC$. What is the product of AI, BI, and CI?

7. [Ray Li] In triangle ABC, $BC = 9$. Points P and Q are located on BC such that $BP = PQ = 2$, $QC = 5$. The circumcircle of APQ cuts AB, AC at D, E respectively. If $BD = CE$, then find $\frac{AB}{AC}$.

8. [CMIMC 2017, Own] In acute triangle ABC, points D and E are the feet of the angle bisector and altitude from A respectively. Suppose that $AC - AB = 36$ and $DC - DB = 24$. Compute $EC - EB$.

9. [OMO 2013, Evan Chen] Let $ABXC$ be a parallelogram. Points K, P, Q lie on $\overline{BC}$ in this order such that $BK = \frac{1}{3}KC$ and $BP = PQ = QC = \frac{1}{3}BC$. Rays XP and XQ meet $\overline{AB}$ and $\overline{AC}$ at D and E, respectively. Suppose that $\overline{AK} \perp \overline{BC}$, $EK - DK = 9$ and $BC = 60$. Find $AB + AC$.

10. [NIMO 11] In triangle ABC, $\sin A \sin B \sin C = \frac{1}{1000}$ and $AB \cdot BC \cdot CA = 1000$. What is the area of triangle ABC?

    (**Hint:** How does $AB \cdot BC \cdot CA$ relate to the area of $\triangle ABC$?)

11. Two circles, ω_1 and ω_2, have radii of 5 and 12 respectively, and their centers are 13 units apart. The circles intersect at two different points P and Q. A line l is drawn through P and intersects the circle ω_1 at $X \neq P$ and ω_2 at $Y \neq P$. Find the maximum value of $PX \cdot PY$.

★ 12. [HMMT 2004] Right triangle XYZ has right angle at Y and $XY = 228$, $YZ = 2004$. Angle Y is trisected, and the angle trisectors intersect XZ at P and Q so that X, P, Q, and Z lie on XZ in that order. Find the value of $(PY + YZ)(QY + XY)$.

★ 13. [CMIMC 2017, Own] Two circles ω_1 and ω_2 are said to be *orthogonal* if they intersect each other at right angles. In other words, for any point P lying on both ω_1 and ω_2, if ℓ_1 is the line tangent to ω_1 at P and ℓ_2 is the line tangent to ω_2 at P, then $\ell_1 \perp \ell_2$. (Two circles which do not intersect are not orthogonal.)

Let $\triangle ABC$ be a triangle with area 20. Orthogonal circles ω_B and ω_C are drawn with ω_B centered at B and ω_C centered at C. Points T_B and T_C are placed on ω_B and ω_C respectively such that AT_B is tangent to ω_B and AT_C is tangent to ω_C. If $AT_B = 7$ and $AT_C = 11$, what is $\tan \angle BAC$?