Set Theory

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Notation

We denote $[n] := \{1, 2, \ldots, n\}$.
If A, B are sets, then:

- $A \subseteq B$ if every element of A is an element of B (we say A is a *subset* of B).
- $A \cup B$ is the set of all elements that are in A, or B, or both (the *union* of A and B).
- $A \cap B$ is the set of all elements that are in both A and B (*intersection* of A, B).

Theoretical Exercises

1. Show that $[n]$ has 2^n subsets. Show that $2^n = \sum_{i=0}^{n} \binom{n}{i}$ by counting the subsets of $[n]$ in a second way.

2. Let e_n denote the number of subsets of $[n]$ with an even number of elements, and o_n denote the number of subsets of $[n]$ with an odd number of elements. Show that $e_n = o_n$ for all n.

3. Prove the identity $\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2$ by showing that the right-hand side equals the number of subsets of $[2n]$ of size n.

4. Prove the 'hockey-stick identity' $\binom{n+1}{k+1} = \sum_{i=0}^{k} \binom{n}{i}$ by describing an appropriate object to count in two ways.

5. (a) Let (A, B) be a pair of sets such that $A \subseteq B \subseteq [n]$. Compute the number of such pairs when $n = 2$ and when $n = 3$.

   (b) Find a formula for the number of such pairs in terms of n, and prove your formula.

   (c) Let $(A_1, A_2, \ldots, A_k)$ be a k-tuple of sets such that $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_k \subseteq [n]$. Find a formula for the number of such pairs in terms of n and k, and prove your formula.

Problems

5. Set A has 20 elements, and set B has 15 elements. What is the smallest possible number of elements in $A \cup B$? [1]

6. Two subsets of the set $S = \{a, b, c, d, e\}$ are to be chosen so that their union is S and their intersection contains exactly two elements. In how many ways can this be done, assuming that the order in which the subsets are chosen does not matter? [2]

[1] AMC 10A 2011, Problem 6)
[2] AMC 10A 2008, Problem 23

7. A game uses a deck of n different cards, where n is an integer and $n \geq 6$. The number of possible sets of 6 cards that can be drawn from the deck is 6 times the number of possible sets of 3 cards that can be drawn. Find n. [3]

8. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. How many (potentially empty) subsets T of S are there such that, for all x, if $x \in T$ and $2x \in S$ then $2x \in T$? [4]

9. Define an ordered triple (A, B, C) of sets to be *minimally intersecting* if $|A \cap B| = |B \cap C| = |C \cap A| = 1$ and $A \cap B \cap C = \emptyset$. For example, $(\{1, 2\}, \{2, 3\}, \{1, 3, 4\})$ is a minimally intersecting triple. Let N be the number of minimally intersecting ordered triples of sets for which each set is a subset of $\{1, 2, 3, 4, 5, 6, 7\}$. Find the remainder when N is divided by 1000. [5]

10. Let $\mathcal{S}$ be the set $\{1, 2, 3, \ldots, 10\}$ Let n be the number of sets of two non-empty disjoint subsets of $\mathcal{S}$. (Disjoint sets are defined as sets that have no common elements.) Find the remainder obtained when n is divided by 1000. [6]

11. Let P be the set of all subsets of [6]. Subsets A and B of [6], not necessarily distinct, are chosen independently and at random from P. Compute the probability that B is contained in at least one of A or $S \backslash A$. (The set $S - A$ is the set of all elements of S which are not in A.) [7]

12. How many subsets of $\{1, 2, \ldots, 10\}$ do not contain two consecutive integers?[8]

13. How many non-empty subsets $S \subseteq \{1, 2, \ldots, 10\}$ satisfy $\max(S) \leq |S| + 2$? Note that $\max(S)$ is defined to be the maximal element of S. [9]

14. Let $S = \{1, 2, 3\}$. How many collections $\mathcal{T}$ of subsets of S have the property that for any two subsets $U \in \mathcal{T}$ and $V \in \mathcal{T}$, both $U \cap V$ and $U \cup V$ are in $\mathcal{T}$? [10]

Challenge Problems

1. Let $A_1, A_2, \ldots, A_k$ be subsets of $[n]$ such that there is no $i, j \in [k]$ such that $A_i \subseteq A_j$. Find, with proof, the maximum possible value of k. [11]

2. Oddtown has n resident, and these residents want to form lots of clubs. Any person can be in as many clubs as they like. The only restrictions are that each club must have an odd number of members, and any two different clubs share an even number of member. Find, with proof, the maximum number of clubs the residents of oddtown can form. [12]

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[3] AIME II 2005, Problem 1
[4] HMMT February 2010, Combinatorics #1
[5] AIME I 2010, Problem 7
[6] AIME II 2002, Problem 9
[7] AIME II 2007, Problem 10
[8] PUMaC 2007, Combinatorics A #4
[9] PUMaC 2012, Combinatorics B #4
[10] HMMT February 2012, Guts #32
[11] Emanuel Sperner (1928)
[12] Famous problem