Complex Numbers
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Problems

If you are unfamiliar with any of the words or symbols below, please refer to the next page for a collection of the relevant information to help you!

1. Compute $|1 + 2i|^2$ and $(1 + 2i)^2$. Do the same for $|2 + 3i|^2, (2 + 3i)^2$. Do you notice anything special about the numbers you find?

2. (F06 NYCIML B19) Compute $(1 - i)^{10}$.

3. (F11 NYCIML B2) Let $z$ be a complex number that satisfies $z + 6i = iz$. Find $z$.

4. (04 AMC 12B #16) A function $f$ is defined by $f(z) = iz$, where $i = \sqrt{-1}$ and $\overline{z}$ is the complex conjugate of $z$. How many values of $z$ satisfy both $|z| = 5$ and $f(z) = z$?

5. (17 AMC 12A #17) There are 24 different complex numbers $z$ such that $z^{24} = 1$. For how many of these is $z^6$ a real number?

6. Using de Moivre’s theorem, find formulas for $\cos 2\theta$, $\cos 3\theta$ in terms of $\cos \theta$.

7. (09 AMC 12A #15) For what value of $n$ is $i + 2i^2 + 3i^3 + \cdots + ni^n = 48 + 49i$?

8. (F09 NYCIML B23) Find the complex number $c$ such that the equation $x^2 + 4x + 6ix + c = 0$ has only one solution.

9. (09 AIME I #2) There is a complex number $z$ with imaginary part 164 and a positive integer $n$ such that $\frac{z}{z+n} = 4i$. Find $n$.

10. (85 AIME #3) Find $c$ if $a, b, c$ are positive integers which satisfy $c = (a + bi)^3 - 107i$. (Hint: 107 is prime.)

11. (94 AIME #8) The points $(0,0), (a,11)$, and $(b,37)$ are the vertices of an equilateral triangle. Find the value of $ab$. (Hint: is there a way to think of points in the plane as complex numbers?)

12. (97 AIME #14) Let $v$ and $w$ be distinct, randomly chosen roots of the equation $z^{1997} - 1 = 0$. Let $\frac{m}{n}$ be the probability that $\sqrt{2} + \sqrt{3} \leq |v + w|$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$. 
Challenge Problems

1. If $p$ is a prime number and $a_0, a_1, \ldots, a_{p-1}$ are rational numbers satisfying

$$a_0 + a_1 \zeta + a_2 \zeta^2 + \cdots + a_{p-1} \zeta^{p-1} = 0,$$

2. (95 IMO) Let $p > 2$ be a prime number and let $A = \{1, 2, \ldots, 2p\}$. Find the number of subsets of $A$ each having $p$ elements and whose sum is divisible by $p$. (Hint: Count the more general $N_k$, the number of subsets of $A$ with $p$ elements whose sum is congruent to $k \mod p$ and consider $p$th roots of unity.)

Background

A complex number is a number of the form $z = a + bi$, where $a, b$ are real numbers and $i$ is the number satisfying $i^2 = -1$. $a$ is referred to as the real part of $z$, denoted by $\text{Re } z$, and $b$ is referred to as the imaginary part, denoted by $\text{Im } z$. Addition and multiplication of imaginary numbers works the same way as multiplication of binomials; that is, if $z = a + bi, w = c + di$, where $a, b, c, d \in \mathbb{R}$ then

$$z + w = (a + bi) + (c + di) = (a + c) + (b + d)i,$$
$$w z w = (a + bi)(c + di) = ac + bdi^2 + i(ad + bc) = (ac - bd) + i(ad + bc).$$

For a complex number $z = a + bi$, we define

- the conjugate of $z$, denoted $\overline{z}$, is $a - bi$;
- the norm or modulus of $z$, denoted $|z|$, is $\sqrt{zz} = \sqrt{a^2 + b^2}$.

de Moivre’s formula

One useful fact is that any complex number $z$ can be written in the form

$$z = |z| \text{cis} \theta,$$

where $0 \leq \theta < 2\pi$, and $\text{cis} \theta = \cos \theta + i \sin \theta$. Often, $\theta$ is referred to as the argument of $z$. Then, de Moivre’s formula states that for any $n \in \mathbb{N}$, $z^n = |z|^n \text{cis}(n\theta)$. This provides an important connection between complex numbers and trigonometry.