1 Introduction

Definition 1 (Euler’s Totient Function). Euler’s Totient Function, denoted $\varphi$, is the number of integers $k$ in the range $1 \leq k \leq n$ such that $\gcd(n, k) = 1$. A closed form of this function is

$$\varphi(n) = n \prod_{\text{prime } p \text{ s.t. } p|n} \left(1 - \frac{1}{p}\right)$$

Property 2 (Multiplicative Property). Euler’s Totient Function satisfies the multiplicative property — that is, for $m, n$ relatively prime, $\varphi(mn) = \varphi(m)\varphi(n)$

Example 3. $\varphi(36) = 36 \ast (1 - \frac{1}{2}) \ast (1 - \frac{1}{3}) = 12$

With each multiplication, we are essentially removing the factors of each prime $p$ from our count. So, the first multiplication removes all the multiples of 2 that are at most 36 (leaving us with the size of $\{1, 3, \cdots, 33, 35\}$), and the second removes the multiples of 3. This can be proved with the Principle of Inclusion-Exclusion.

Definition 4 (Euler’s Totient Theorem). For all non-zero integers $a$ relatively prime to $n$,

$$a^{\varphi(n)} \equiv 1 \pmod{n}$$

Definition 5 (Fermat’s Little Theorem). For any integer $a$ and prime $p$, $a^p \equiv a \pmod{p}$. If $a$ is not a multiple of $p$, this is equivalent to $a^{p-1} \equiv 1 \pmod{p}$. Otherwise, if $a$ is a multiple of $p$, then $a^{p-1} \equiv 0 \pmod{p}$.

2 Problems

1. Compute $\varphi(30)$ and $\varphi(84)$.
2. Let $p, q$ be primes. Can you find a formula for $\varphi(pq)$? Using this, compute $\varphi(1717)$.
3. Compute $7^{24} \pmod{15}$ and $55^{49} \pmod{84}$.
4. Find all primes $p$ such that $p$ divides $2^p + 1$. \cite{1}
5. Find the remainder when $4^{1996} + 5^{1997}$ is divided by 9.
6. Find the first positive integer $n$ such that $43^n \equiv 1 \pmod{24}$.

\cite{1}From Number Theory for Mathematical Contests by David A. Santos
7. One of Euler’s conjectures was disproved when it was found that there was a positive integer such that \(133^5 + 110^5 + 84^5 + 27^5 = n^5\). Find \(n\). \[2\]

8. Consider the sequence \(a_1, a_2, \ldots\) defined by

\[a_n = 2^n + 3^n + 6^n - 1 \quad (n = 1, 2, \ldots)\]

Determine all positive integers that are relatively prime to every term of the sequence. \[3\]

9. Prove that \(252 \mid n^9 - n^3\). \[1\]

10. Prove that there exists a positive integer \(k\) such that \(k \cdot 2^n + 1\) is composite for all integers \(n\).

3 Challenge Problems

1. Prove that for any \(m \in \mathbb{N}\), the sequence \(2, 2^2, 2^{2^2}, 2^{2^{2^2}}, \ldots \) (mod \(m\)) is constant from a certain point on. \[4\]

2. Let \(\gcd(m, n) = 1\). Prove that \(m^{\varphi(n)} + n^{\varphi(n)} \equiv 1 \pmod{mn}\). \[1\]

3. Find all natural numbers \(n\) that divide \(1^n + 2^n + \cdots + (n-1)^n\). \[1\]

4. Let \(n\) be a positive integer such that \(n + 1\) is divisible by 24. Prove that the sum of all the divisors of \(n\) is divisible by 24. \[5\]

5. Find all positive integer solutions to \(3^x + 4^y = 5^z\). \[6\]