Modular Arithmetic and Divisibility

Number Theory

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1 Introduction

Definition 1 (Divisibility)**.** An integer a is said to be divisible by some nonzero integer b if there exists an integer c such that $a = bc$. Alternatively, for $b \neq 0$, $\frac{a}{b}$ is an integer.

Definition 2 (Euclidean Algorithm)**.** Given integers a, b, the series of divisors $q_1, q_2, \ldots$ such that $a = bq_1 + q_2, b = q_2q_3 + q_4, q_2 = q_4q_5 + q_6, \ldots$ (see example). The final value (when the other is 0) gives $\gcd(a, b)$, i.e. the greatest common divisor of a and b.

Example 3. Find $\gcd(126, 224)$.

Solution.

$$\begin{aligned}
\gcd(126,224) &= \gcd(126, 224-126) && 224 = 1 \times 126 + 98 \\
\gcd(126,98) &= \gcd(98, 126-98) && 126 = 1 \times 98 + 28 \\
\gcd(98,28) &= \gcd(28, 98 - 3\cdot 28) && 98 = 3 \times 28 + 14 \\
\gcd(28,14) &= \gcd(14, 28 - 2\cdot 14) && 28 = 2 \times 14 + 0 \\
&= \gcd(14, 0)
\end{aligned}$$

Thus, $\gcd(126, 224) = \boxed{14}$. □

Definition 4 (Relatively Prime)**.** Given integers a, b, they are called relatively prime or coprime if $\gcd(a, b) = 1$.

Definition 5 (Prime)**.** An integer p is called prime if when p divides a product ab (where a, b are integers), then p divides a or p divides b. Equivalently p is prime if $p = ab$ (where a, b are integers), then either $a = 1$ or $b = 1$.

Theorem 6 (Fundamental Theorem of Arithmetic)**.** Every nonzero integer can be written uniquely (up to order) as a product of primes.

Definition 7 (Modular Arithmetic)**.** Given integers a, b, c, $b \neq 0$, $a \equiv c \pmod{b}$ if b divides $(a - c)$.

Example 8.

$5 \equiv 2 \pmod 3$	$17 \equiv 12068357 \pmod{10}$	$54 \equiv 42 \equiv 0 \pmod 6$
$2 \not\equiv 1 \pmod 3$	$4 + 1 \equiv 29 + 6 \pmod 5$	$3 \times -1 \equiv 19 \times 15 \pmod 8$

2 Problems

1. Using modular arithmetic, show that 3 divides n if and only if 3 divides the sum of the digits of n. Do the same for 9. Can you find something similar for 11?

2. Find $\gcd(221, 299)$ and $\gcd(2520, 399)$.

3. Compute the remainder when $333 + 999$ and 3333×7777 are divided by 5.

4. How many steps does it take the Euclidean Algorithm to reach $(1, 0)$ when the input is $(n + 1, n)$?[1]

5. Let n be a positive integer. Construct a set of n consecutive positive integers that are not prime.[1]

6. Find all positive integers n such that $(n + 1)$ divides $(n^2 + 1)$.[2]

7. Find all primes in the form $n^3 - 1$.[2]

8. What is the largest positive integer n for which $(n + 10)$ divides $n^3 + 100$?[3]

9. Show that $\underbrace{1 \dots 1}_{\text{91 ones}}$ is composite.[2]

10. A year is a leap year if and only if the year number is divisible by 400 (such as 2000) or is divisible by 4 but not 100 (such as 2012). The 200$^{\text{th}}$ anniversary of the birth of novelist Charles Dickens was celebrated on February 7, 2012, a Tuesday. On what day of the week was Dickens born?[4]

11. What is the largest prime factor of 7999488?[5]

12. An n-digit number is cute if its n digits are an arrangement of the set $\{1, 2, \dots, n\}$ and its first k digits form an integer that is divisible by k, for $k = 1, 2, \dots, n$. For example, 321 is a cute 3-digit integer because 1 divides 3, 2 divides 32 and 3 divides 321. How many cute 6-digit numbers are there?[6]

13. An old receipt has faded. It reads 88 chickens at the total of $\$x4.2y$, where x and y are unreadable digits. How much did each chicken cost?[2]

14. Find the smallest positive integer such that $\frac{n}{2}$ is a square and $\frac{n}{3}$ is a cube.[2]

15. How many primes have alternating 1s and 0s in base 10 (like 101)?[7]

16. If $a, b \in \mathbb{N}$ such that $\frac{a}{b} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1318} + \frac{1}{1319}$, show that 1979 divides a.[8]

[1] From *Mathematical Thinking* by John P. D'Angelo and Douglas B. West
[2] From *Number Theory for Mathematical Contests* by David A. Santos
[3] From AIME 1986
[4] AMC 10/12A 2012
[5] PUMaC 2011
[6] AHSME 1991
[7] Putnam 1989
[8] IMO 1979